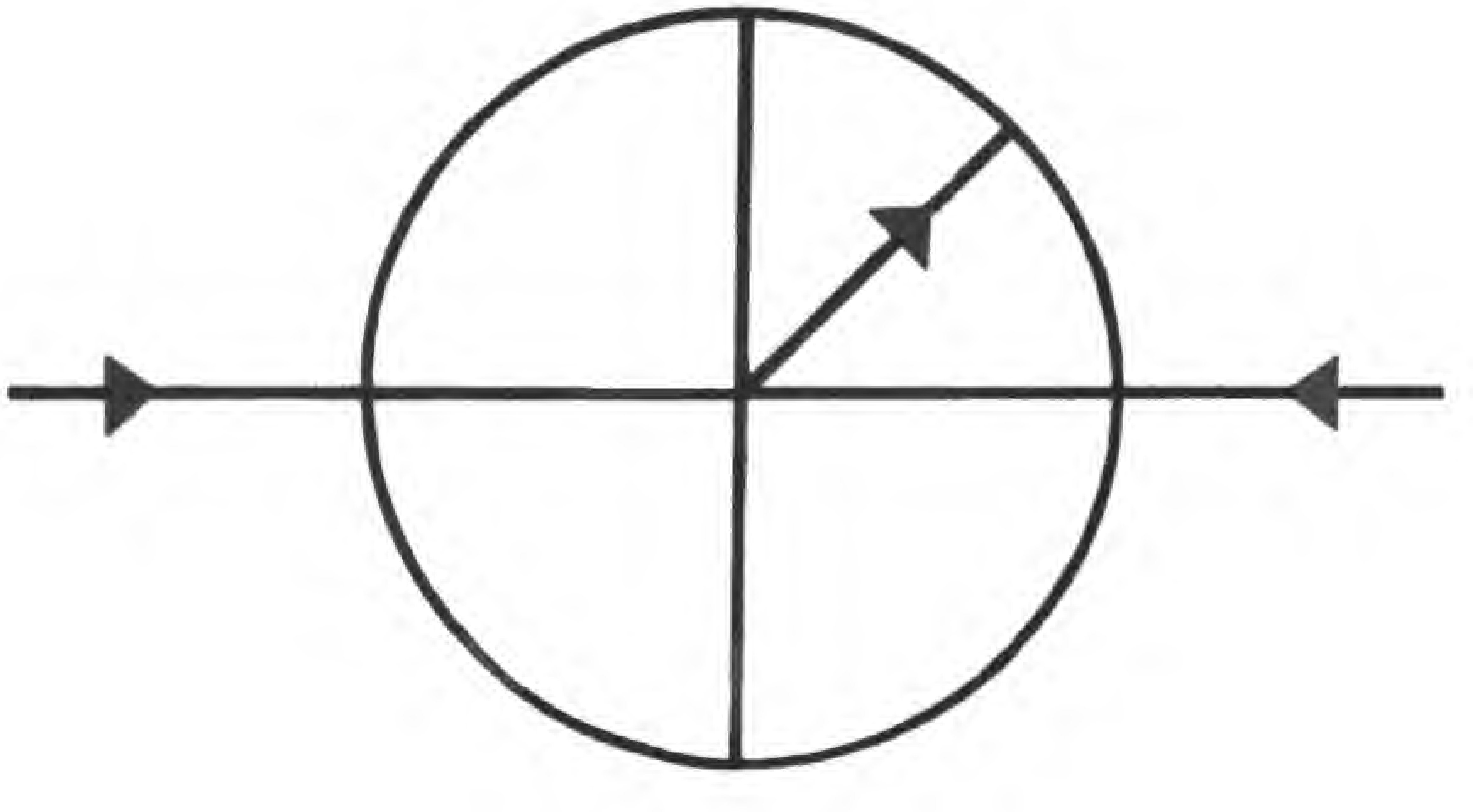
METHODS OF STRUCTURAL ANALYSIS



Negussie Tebedge

Methods of Structural Analysis provides the student of engineering with a concise working description of the classical methods of structural analysis and introduces the concept of matrix formulations of structures.

The basic principles of structural analysis are brought out by a simplified, coherent approach aided by the use of numerous diagrams and worked examples.

Students undertaking courses in the theory of structures and structural analysis will find this book extremely useful either as a main text, or as a supplement to other works in the field.

For a note on the author, please see the back flap.

ISBN 0 333 35093 6

METHODS
OF
STRUCTURAL ANALYSIS

This book is published with the financial support of the African Network of Scientific and Technical Institutions (ANSTI), an organisation within UNESCO.

Methods of Structural Analysis

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ANSTI

First published 1983 by
THE MACMILLAN PRESS LTD
London and Basingstoke
Companies and representatives throughout the world.

ISBN 0 333 35093 6 hardcover
ISBN 0 333 35292 0 paperback

Typeset by
STYLESET LIMITED
Salisbury, Wiltshire

Printed in Hong Kong

To my parents

Contents

Preface

This textbook has been compiled from a set of lecture notes developed while teaching courses in the theory of structures to civil engineering students at Addis Ababa University during the past seven years. The book is primarily intended for use as a text for instruction and contains sufficient material for a two-semester course in theory of structures. It may also be useful to the structural engineer who wishes to strengthen his background in structural mechanics.

The purpose of this book is to present a balanced treatment of the fundamental principles of structural mechanics, with their applications to the analysis of structural systems and their components. The coverage is selective, to allow a thorough treatment of the most common and useful analytical methods of structural analysis.

An attempt is made to present the subject matter in a unified, coherent and easy-to-understand manner which brings out the basic principles underlying the field of structural theory. The book is illustrated with ample example problems, to which solutions are presented to demonstrate the various methods, and also to widen the scope of the subject covered by the text.

The author is indebted to the authors of the many books he has freely consulted in the preparation of this work. The author also wishes to acknowledge his debt to all his students who, over the years, checked out the examples and assignment problems.

NEGUSSIE TEBEDGE
Addis Ababa
June, 1982

1. Introduction

1.1 STRUCTURAL ANALYSIS

Structural analysis is the process of determining the response of a structure due to specified loadings in order to satisfy essential requirements of function, safety, economy and sometimes aesthetics. This response is usually measured by calculating the reactions, internal forces of members, and displacements of the structures.

Structures may be classified into two general categories: ***statically determinate*** and ***statically indeterminate***. A structure which can be completely analysed by means of statics alone is called statically determinate. It then follows that a statically indeterminate structure is one which cannot be analysed by means of statics alone.

There are specific advantages and disadvantages in using one type of structure over the other. The primary advantage of a statically indeterminate structure is that it will generally have lower bending moment and shear force than a comparable determinate structure. Another advantage of a statically indeterminate structure is that it is generally stiffer for a given weight of material than a comparable determinate structure. Both of these advantages are a result of continuity of structural members acting to reduce stress intensities and displacements. A statically indeterminate structure can often furnish a compensation by redistribution within the structure in the case of overloads. On the other hand, however, indeterminate structures introduce computational difficulty in establishing the required equations. Another disadvantage is that indeterminate structures are, in normal cases, internally stressed due to differential settlement of supports, temperature changes and errors in the fabrication of members.

1.2 STATICAL INDETERMINACY

Consider a structure in space subjected to non-coplanar system forces. For the structure to be in equilibrium, the components of the resultants in the three

orthogonal directions must vanish. This condition constitutes the six equations of equilibrium in space which are written as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0$$
$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \qquad [1.1]$$

For a structure subjected to a coplanar force system, only three of the six equations of equilibrium are applicable. The three equations of equilibrium in the xy plane are

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0 \qquad [1.2]$$
$$\Sigma M_z = 0$$

When a structure is in equilibrium, each member, joint, or segment of the structure must also be in equilibrium and the equations of equilibrium must also be satisfied. As discussed earlier, a structure which can be analysed by means of the equations of equilibrium alone is statically determinate. This book deals with statically indeterminate structures in which the structures cannot be analysed by the equations of equilibrium alone.

When a structure is statically indeterminate, there is some freedom of choice in selecting the member or reaction to be regarded as redundant. When the reaction is taken as the redundant, the structure is said to be *externally indeterminate*. On the other hand, when the member itself is regarded as the redundant, the structure is said to be ***internally indeterminate***. It is also possible that the structure may have a combination of external and internal indeterminacy.

The question of identifying external or internal indeterminacy is largely of academic interest. What is of primary importance in the analysis of indeterminate structures is to know the degree of total indeterminacy. Nevertheless, a separate discussion of external and internal indeterminacy is desirable as a method to evaluate the degree of total indeterminacy.

(a) External Indeterminacy If the total number of reactions in a structure exceeds the number of the equations of equilibrium applicable to the structure, the structure is said to be externally indeterminate. The structures shown in Fig. 1.1 are examples of external indeterminacy. Each of the structures has five reaction components. Since there are only three equations of equilibrium, there are two extra reaction components that cannot be determined by statics. The number of unknown reactions in excess of the applicable equations of equilibrium defines the degree of indeterminacy. Thus the structures of Fig. 1.1 are indeterminate to the second degree. An alternative approach to determine the degree of indeterminacy would be to remove selected redundant reactions until the structure is reduced to a statically determinate and stable base or primary structure.

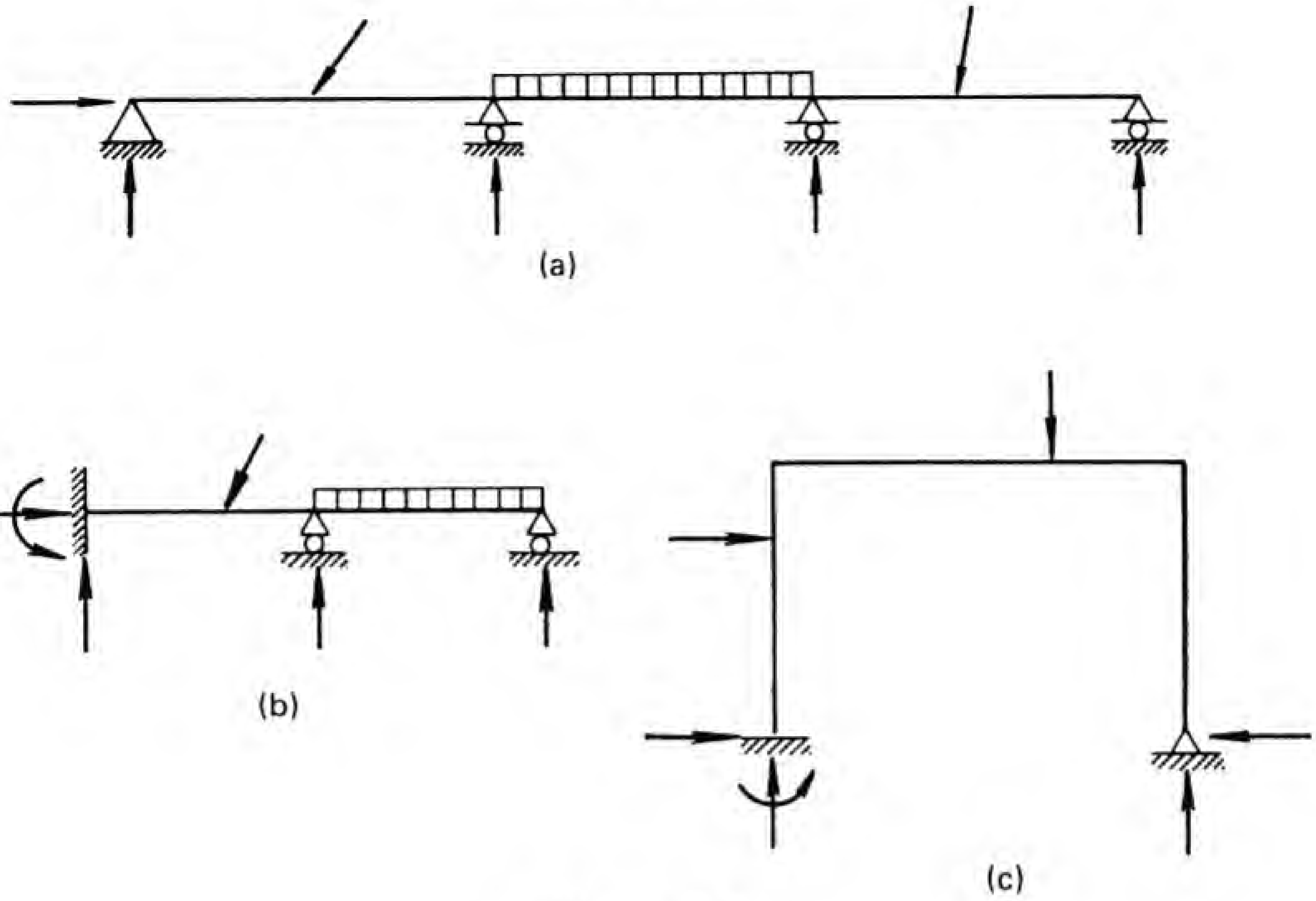


Figure 1.1

(b) Internal Indeterminacy A structure is internally indeterminate when it is not possible to determine all internal forces by using the three equations of static equilibrium. For the great majority of structures, the equation of whether or not they are indeterminate can be decided by inspection. For certain structures this is not so, and for these types rules have to be established. The internal indeterminacy of trusses will be first considered, and then that of continuous frames.

It is evident that any truss developed by using three bars connected at three joints in the form of a hinged triangle, and then using two bars to connect each additional joint, forms a stable and determinate truss. This is because the shape of the triangle cannot be changed without changing the length of any of the members. For stable and determinate trusses, built up as an assemblage of triangles, there are two conditions of equilibrium for each joint, so that if there are j joints, m members and r reaction components, a test for statical determinacy is:

$$2j = m + r \qquad [1.3]$$

In this equation, the left-hand side represents the total possible number of equations of equilibrium, while the right-hand side represents the total number of unknown forces.

The above equation is usually written in the form

$$m = 2j - r \qquad [1.4]$$

If there are more members than are indicated by the equation, then the

structure is statically indeterminate; whereas if it has fewer members it is unstable. Caution must be exercised in applying the above equation because of the fact that the fulfilment of this equation is a *necessary* condition but not *sufficient* for internal stability of trusses. This may be summarised as

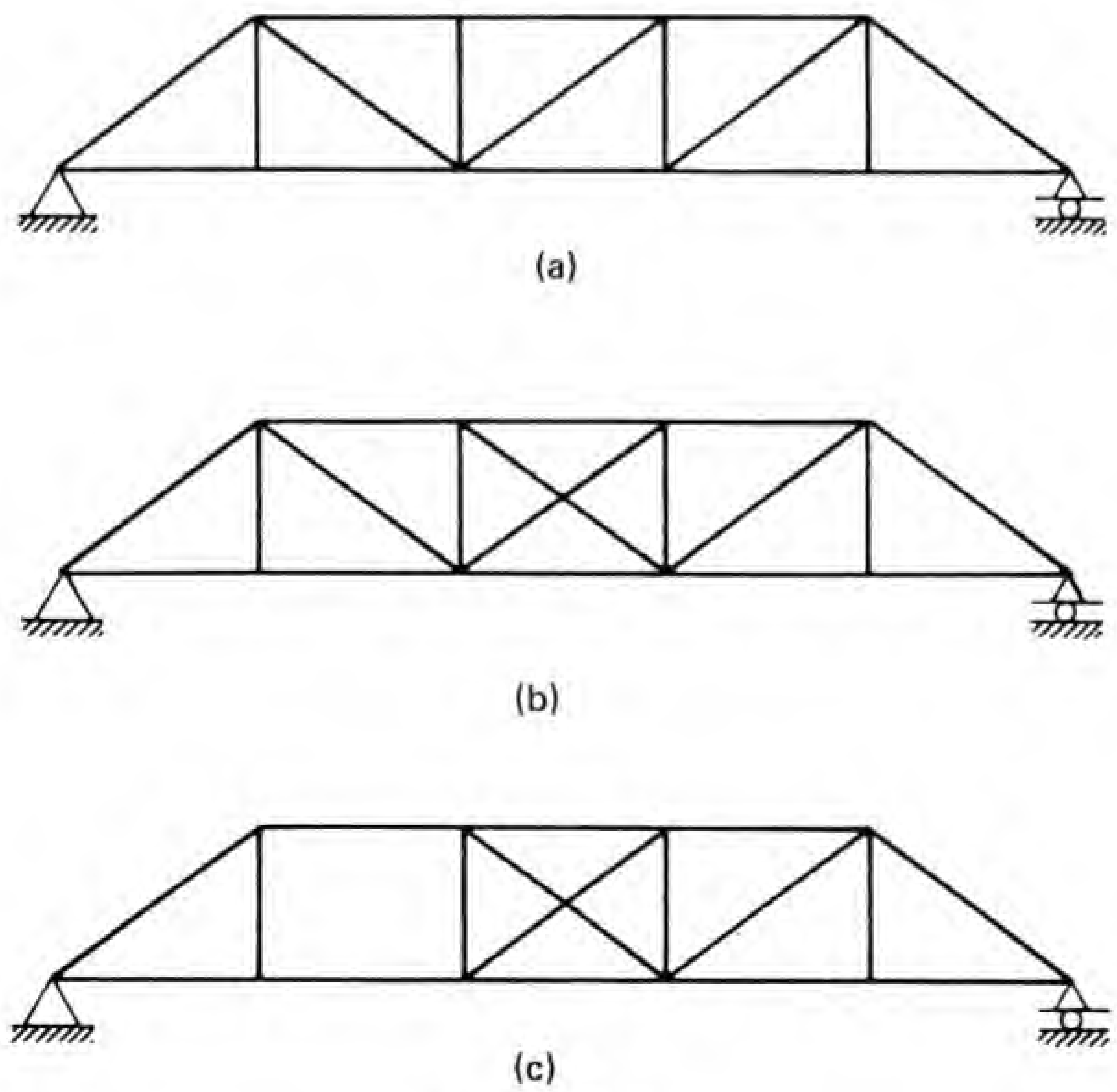
$m = 2j - r$ (determinate if stable)

$m > 2j - r$ (indeterminate if stable)

$m < 2j - r$ (unstable)

The truss in Fig. 1.2(a) has $m = 17, j = 10$ and $r = 3$. Application of [1.4] gives $(10 \times 2) - 3 = 17$ members, thus the structure is statically determinate. Referring to Fig. 1.2(b), there are 18 members, or one more member than is needed for a determinate structure; thus, the additional diagonal member is redundant and the truss is indeterminate to the first degree. Figure 1.2(c) represents the omission of one diagonal member, keeping the same total number of bars, $m = 17$. Again the condition equation is satisfied. However, inspection of the truss indicates that the structure is unstable with one panel free to collapse, thus causing the entire truss to collapse. Hence, satisfaction of the above equation is not a sufficient condition for internal stability of trusses. Inspection of the structure and consideration of stress paths are more reliable approaches to settle the question of stability and internal indeterminateness of trusses.

An alternative approach to determine the degree of indeterminacy of trusses



(a)

(b)

(c)

Figure 1.2

is by removing the redundant quantities until a determinate and stable base structure remains.

The number of rigidly jointed frames are subject to shearing forces, bending moment and axial force, so that there are three unknown internal forces for each member, or a total of $3m$ unknown components. Moreover, at each joint three equations of equilibrium can be written, giving $3j$ equations in all. Therefore for a statical determinacy, it is necessary that

$$3j = 3m + r \qquad [1.5]$$

or that the number of redundants n is given by

$$n = 3m + r - 3j$$

When there is a roller or pin support, the degree of indeterminacy is reduced by one or two, respectively, for each support.

An alternative approach, which in this case may be considered more instructive, is the method by inspection where the structure is cut until it becomes a determinate and stable base structure. Consequently, the total number of released internal force components corresponds to the degree of indeterminacy.

1.3 KINEMATIC INDETERMINACY

When a structure is subjected to a system of forces, the overall behaviour of the members of the structure may be defined by the displacement of the joints. The joints undergo displacements in the form of translation and rotation. A system of joint displacements is known to be independent if each displacement can be varied arbitrarily and independently of the other displacements. The number of independent joint displacements that serve to describe all possible displacements of a structure is known as the number of ***degrees of freedom*** or ***degree of kinematic indeterminacy***.

In determining the degree of kinematic indeterminacy, attention is focused on the number of independent displacement degrees of freedom that the structure possesses. If a structure has n degrees of freedom, that is, n number of independent displacement quantities required to describe all possible displacements for any loading condition, the structure is said to be kinematically indeterminate to the nth degree. When these displacements are set to zero, the structure then becomes ***kinematically determinate***.

Consider, for example, the rigid-jointed plane frame shown in Fig. 1.3, which is fixed at supports A and C and has a hinged support at D. Assuming that the axial deformations are negligible, there will be no axial displacements in the frame and the only unknown displacements are the joint rotations θ_B and θ_D at joints B and D, respectively. Since these displacements are independent of one another, the degree of kinematic indeterminacy of this structure is two.

It is observed that the degree of statical indeterminacy of the frame of Fig. 1.3 is four since there are a total of seven possible unknown reactions and three equations of equilibrium. If, for instance, the fixed support at C is replaced by a hinge, the degree of statical indeterminacy is reduced to three since an additional equilibrium condition is introduced. However, the kinematic indeterminacy is increased by one since an independent rotation at C now becomes possible. In general, an introduction of a displacement release decreases the statical indeterminacy and increases the kinematic indeterminacy.

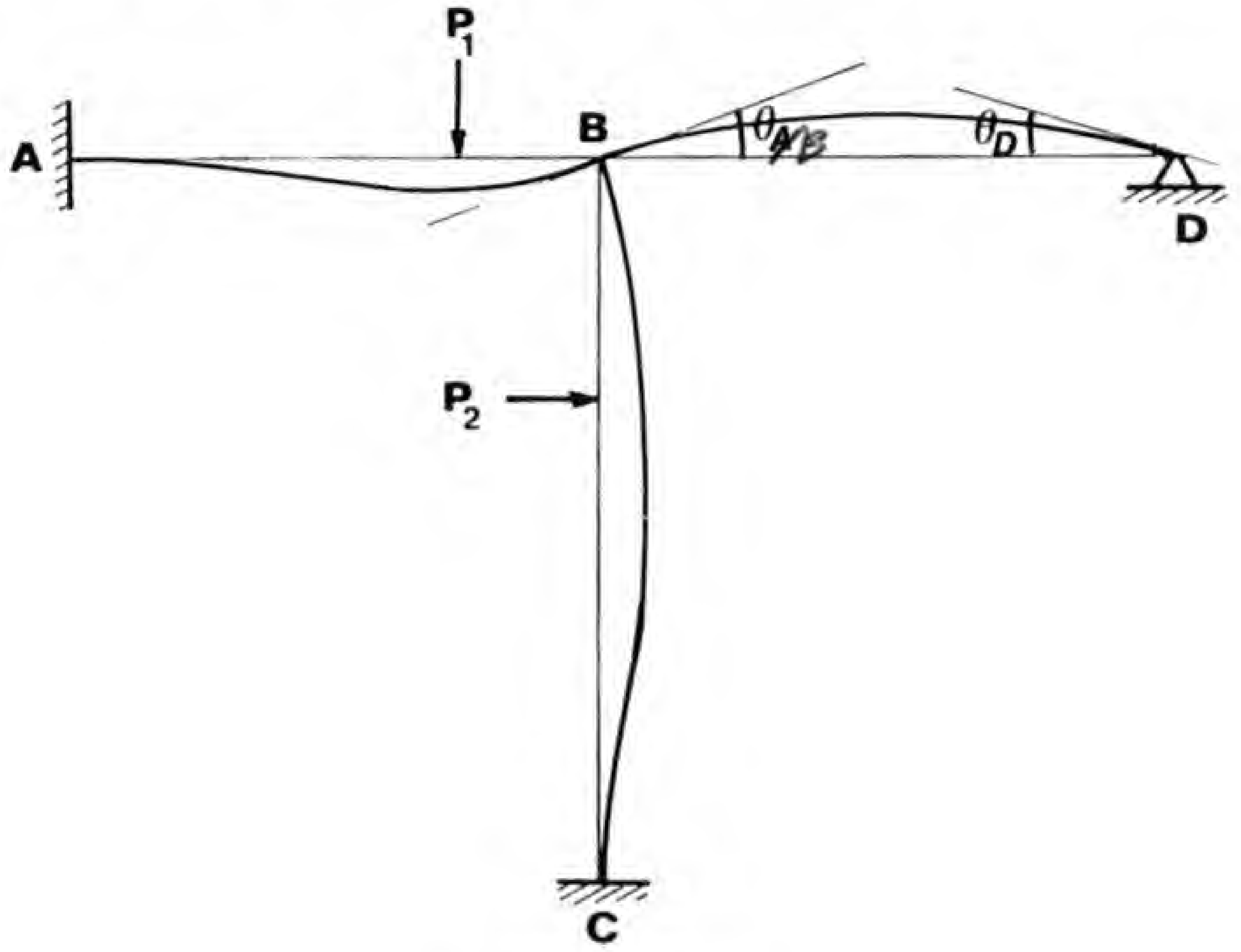


Figure 1.3

1.4 METHODS OF STRUCTURAL ANALYSIS

The objective of structural analysis is to study the response of a structure to specified loadings after determining the external reactions and internal stress resultants. The forces determined must satisfy the conditions of equilibrium and the displacements produced by these forces must be compatible with the continuity of the structure and the support conditions. In determining the unknown forces in a statically indeterminate structure, the equations of equilibrium are not sufficient, and additional equations must be formulated based on compatibility of displacements. These supplementary equations that ensure the compatibility of the displacements with the geometry of the structure are known as the *compatibility* conditions.

Two general methods of analysis are available for the solution of statically indeterminate structures. The first is the *force* or *flexibility* method. This method is simple and conceptually straightforward to understand and provides

an effective method for certain types of structures. In this method the structure is made statically determinate by providing a sufficient number of releases by removing the redundant forces. Due to the given loading condition the primary structure undergoes inconsistency in geometry which must then be corrected by applying the redundant forces such that compatibility conditions throughout the structure are established. This method is sometimes referred to as the *compatibility* method.

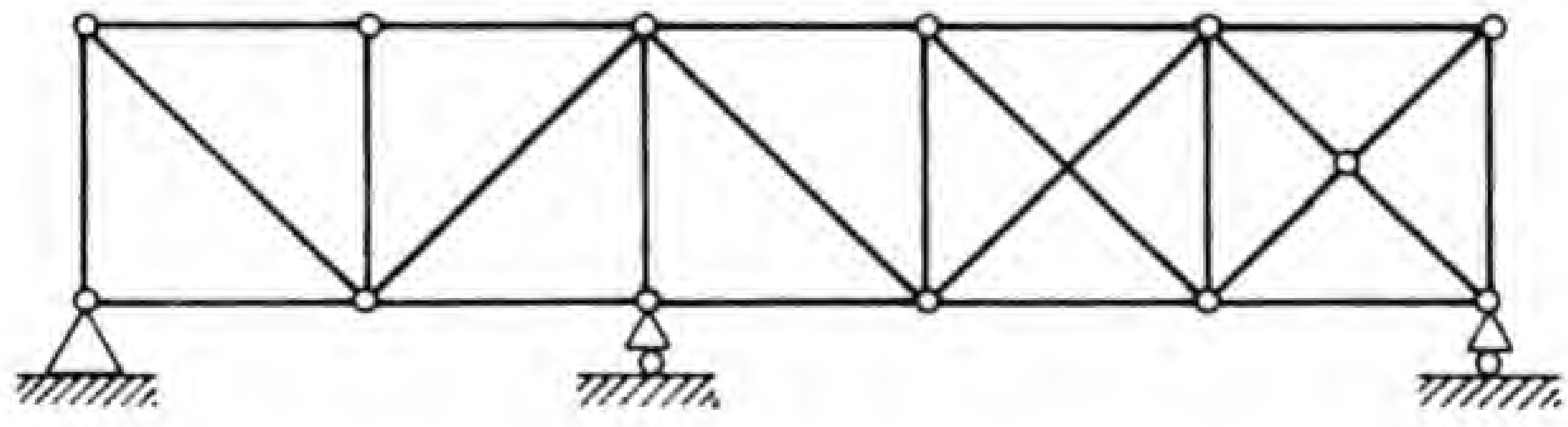
The second method of analysis of statically indeterminate structures is the *displacement* or *stiffness* method. This method is also simple and straightforward and provides an effective method for certain classes of structure. In this method, restraints are imposed to prevent displacement of joints until the structure becomes kinematically determinate and the forces required to produce the restraints are evaluated. Displacements are then permitted to take place at the restrained joints until the imposed restraining forces have been removed such that equilibrium conditions throughout the structure are established. This method is also known as the *equilibrium* method.

Either the force or the displacement method can be used to analyse any structure. The choice of the method of analysis, either force or displacement, depends largely on the degree of statical or kinematic indeterminacy. In both methods, the analysis generally involves the solution of a system of simultaneous equations where the number of unknown variables must be equal to the degree of indeterminacy. If manual calculations are to be adopted, it would be logical to use the method that produces the smaller set of simultaneous equations.

1.5 PROBLEMS

Determine the degree of statical indeterminacy of the structures shown below.

1.1



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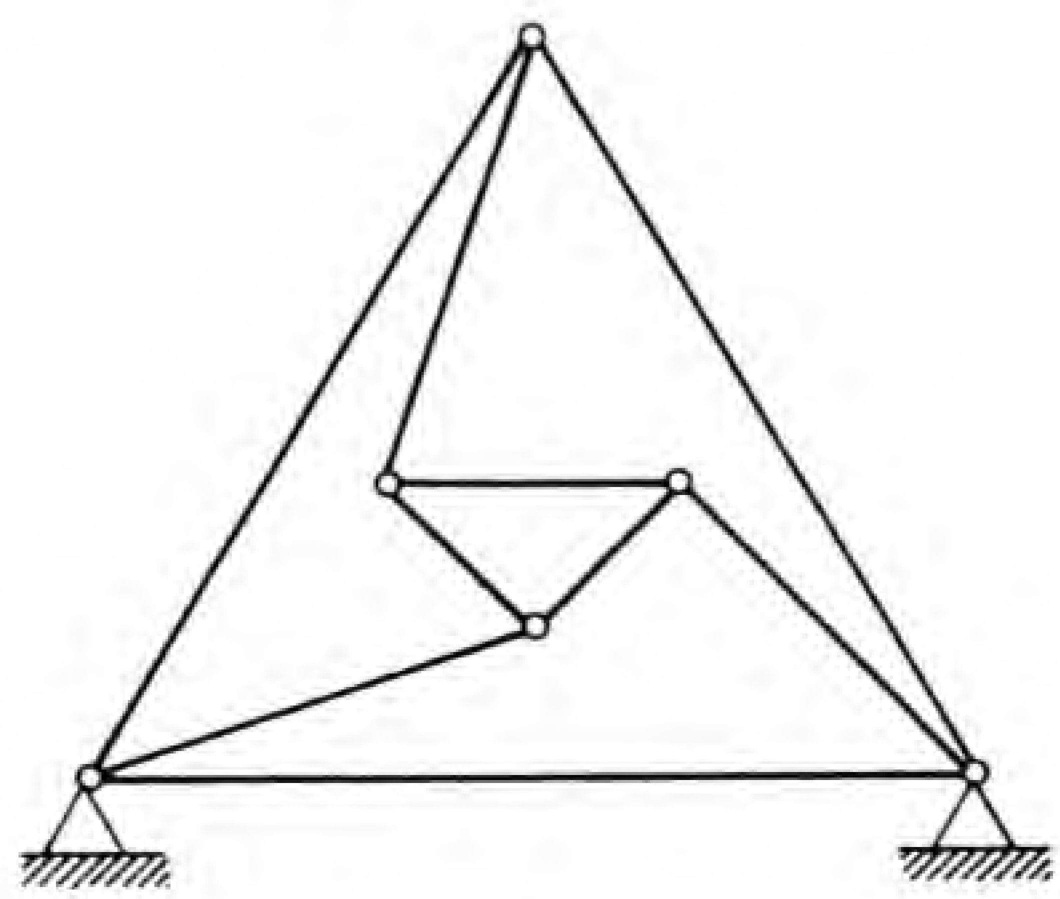

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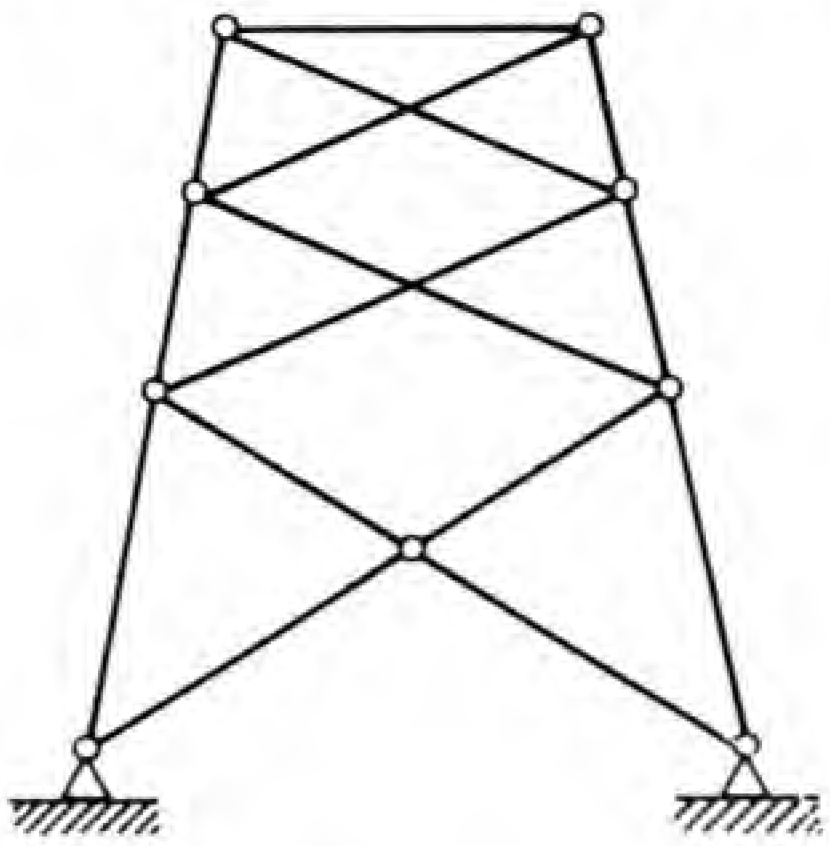

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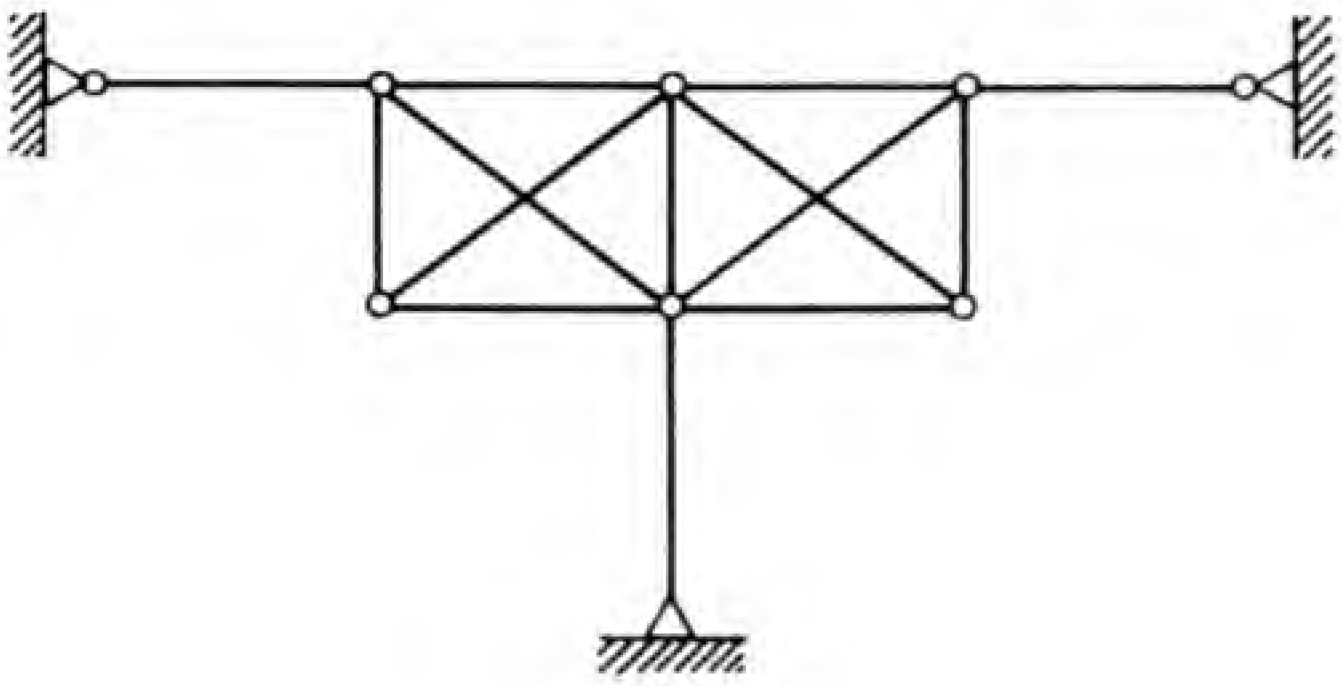

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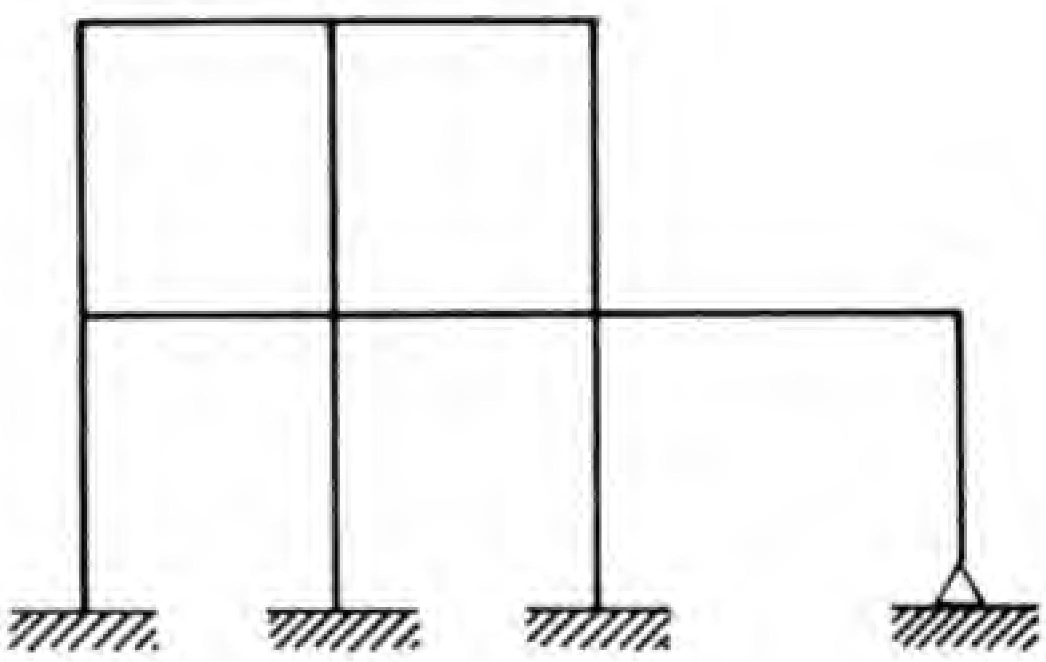

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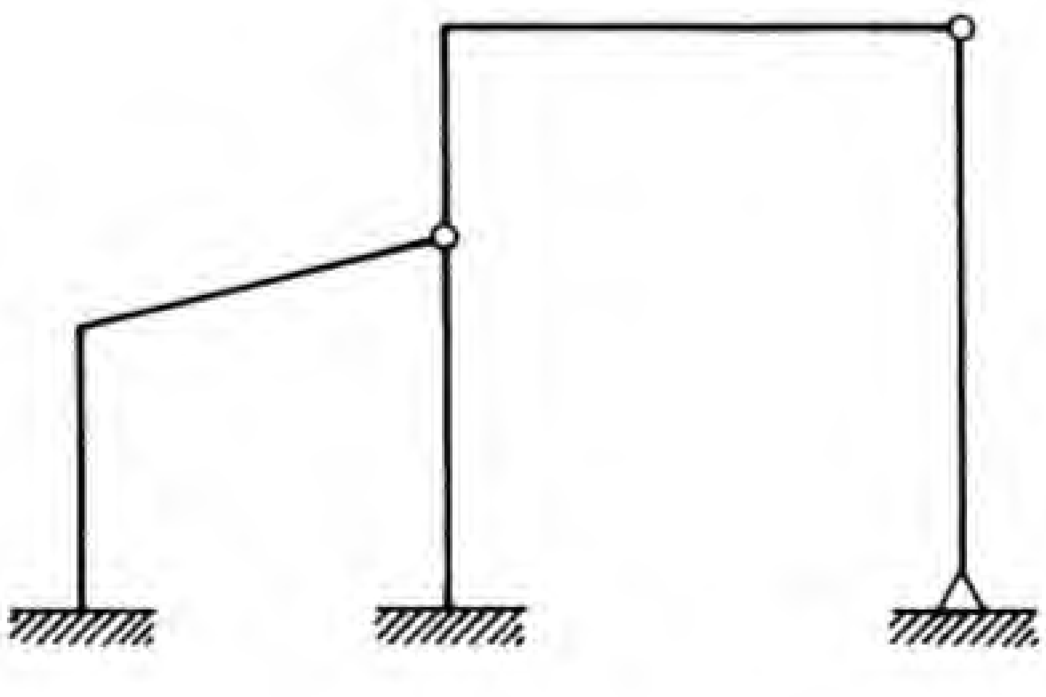

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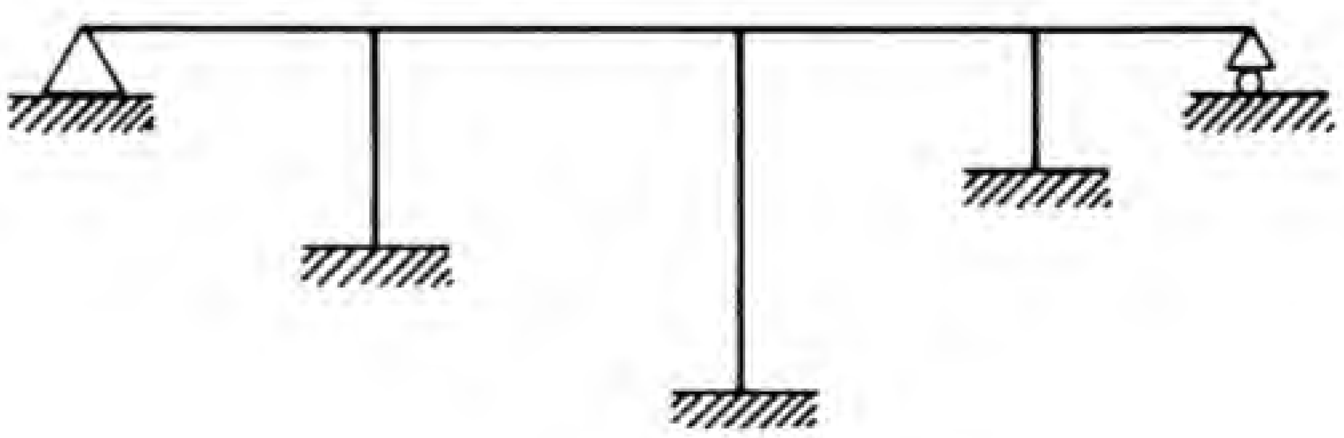
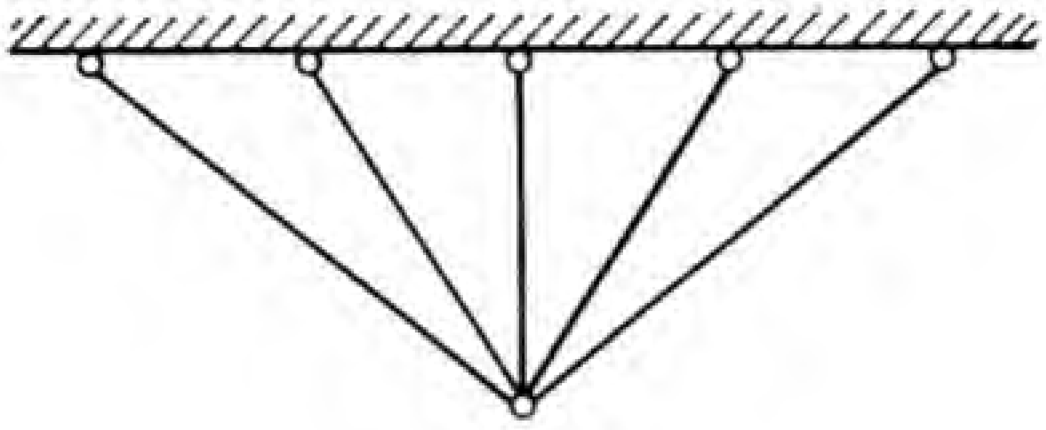


Determine the degree of statical and kinematical indeterminacy of the structures shown below.

1.8



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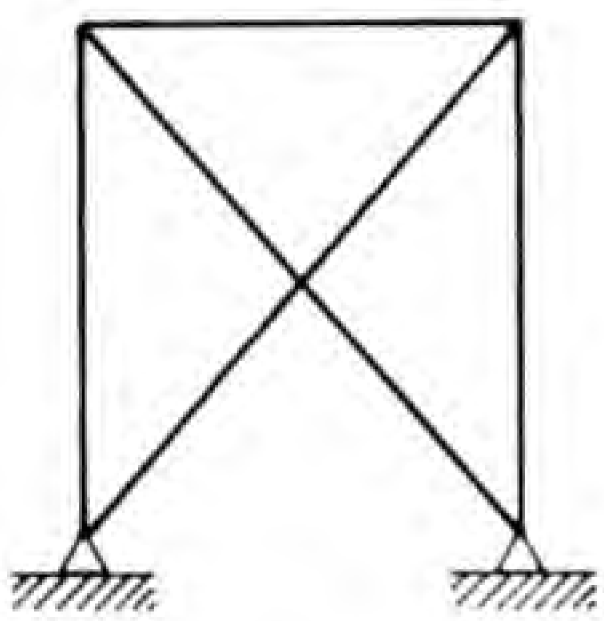

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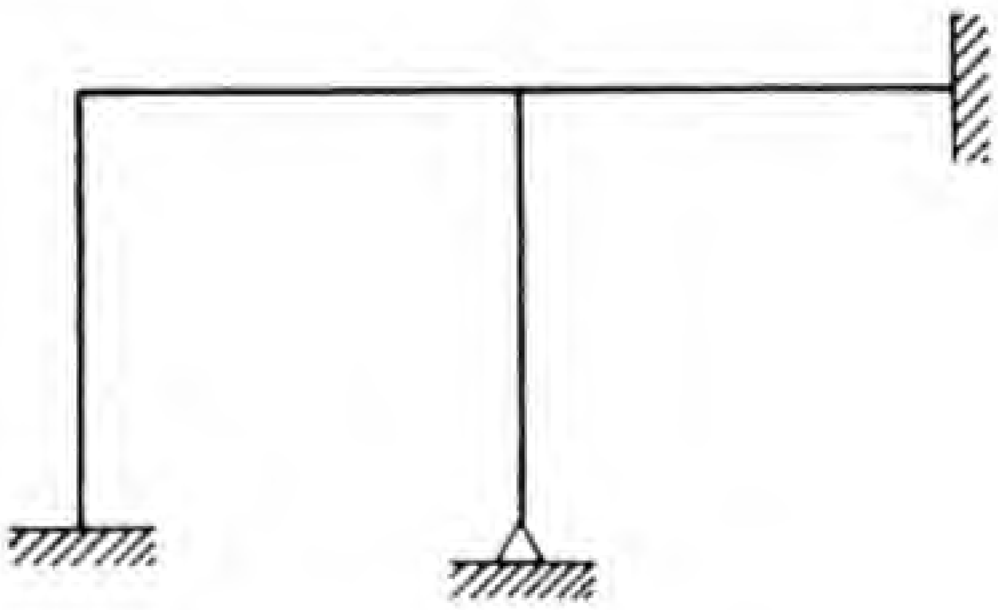

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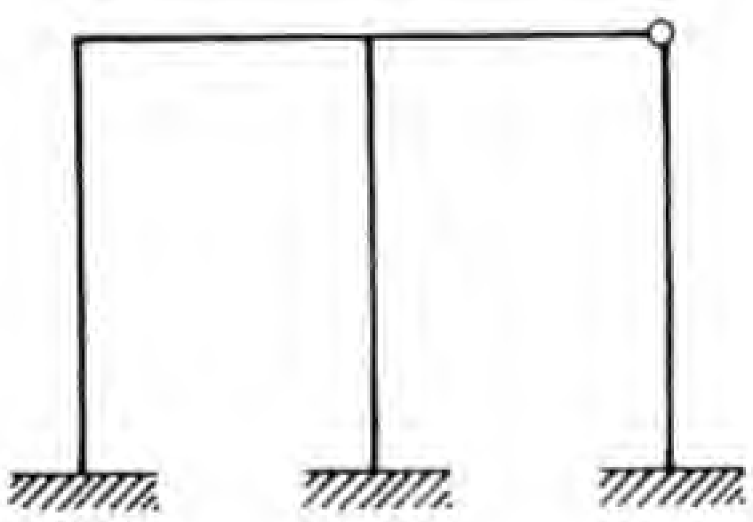

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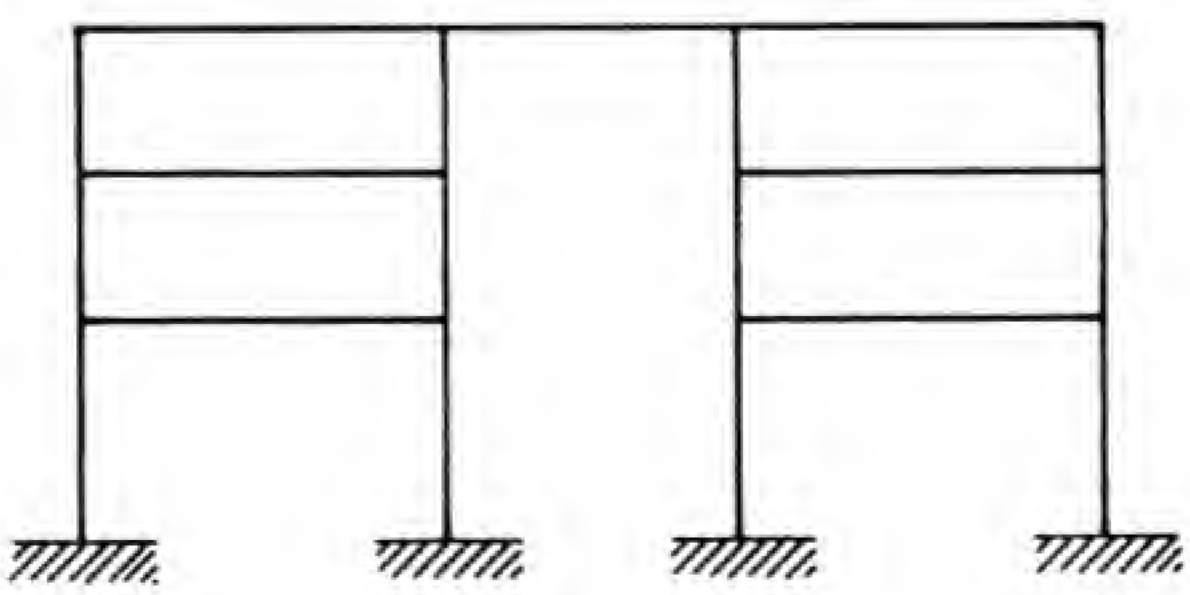

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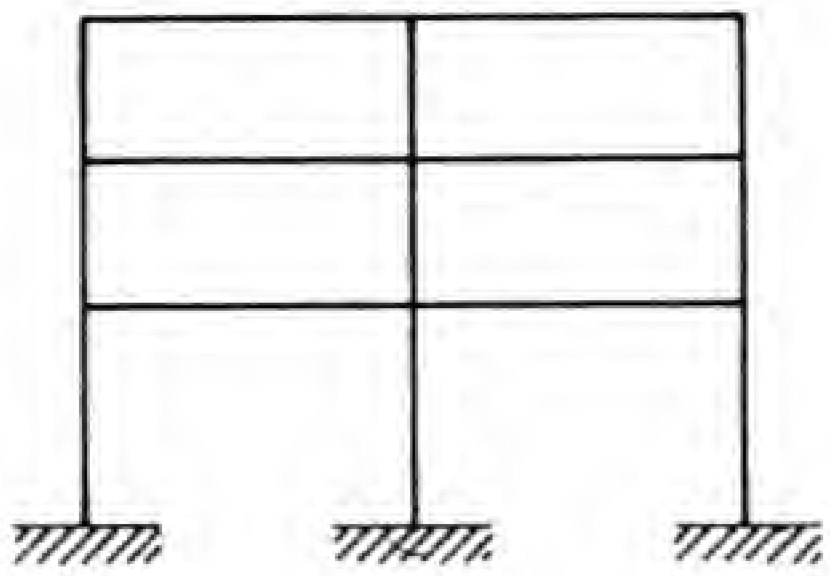

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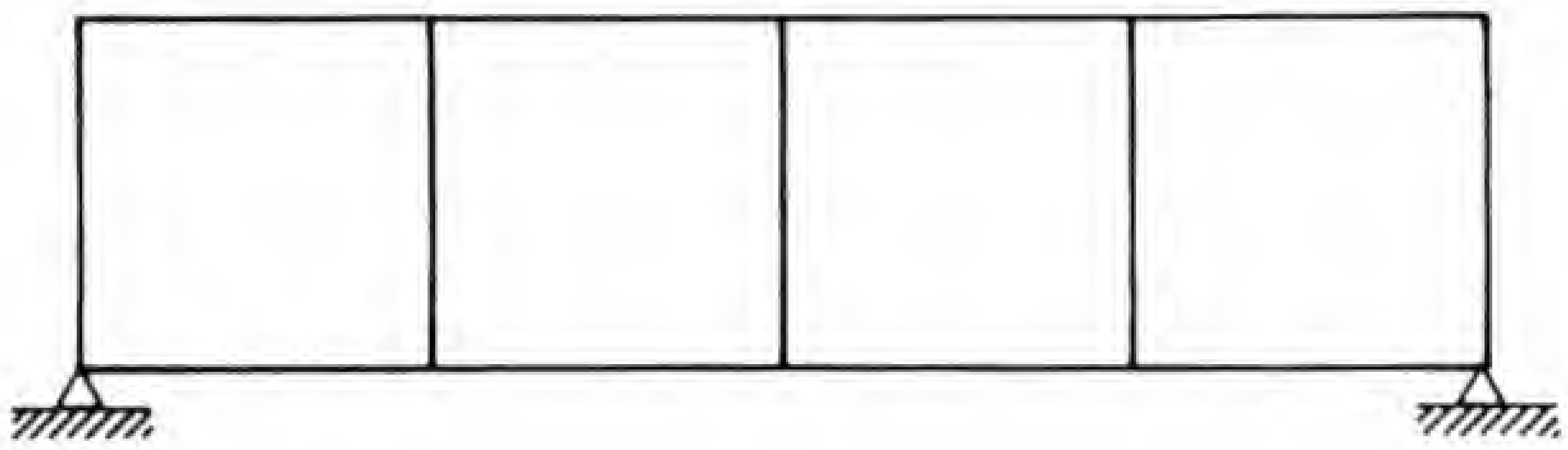


2. Methods of Consistent Displacements

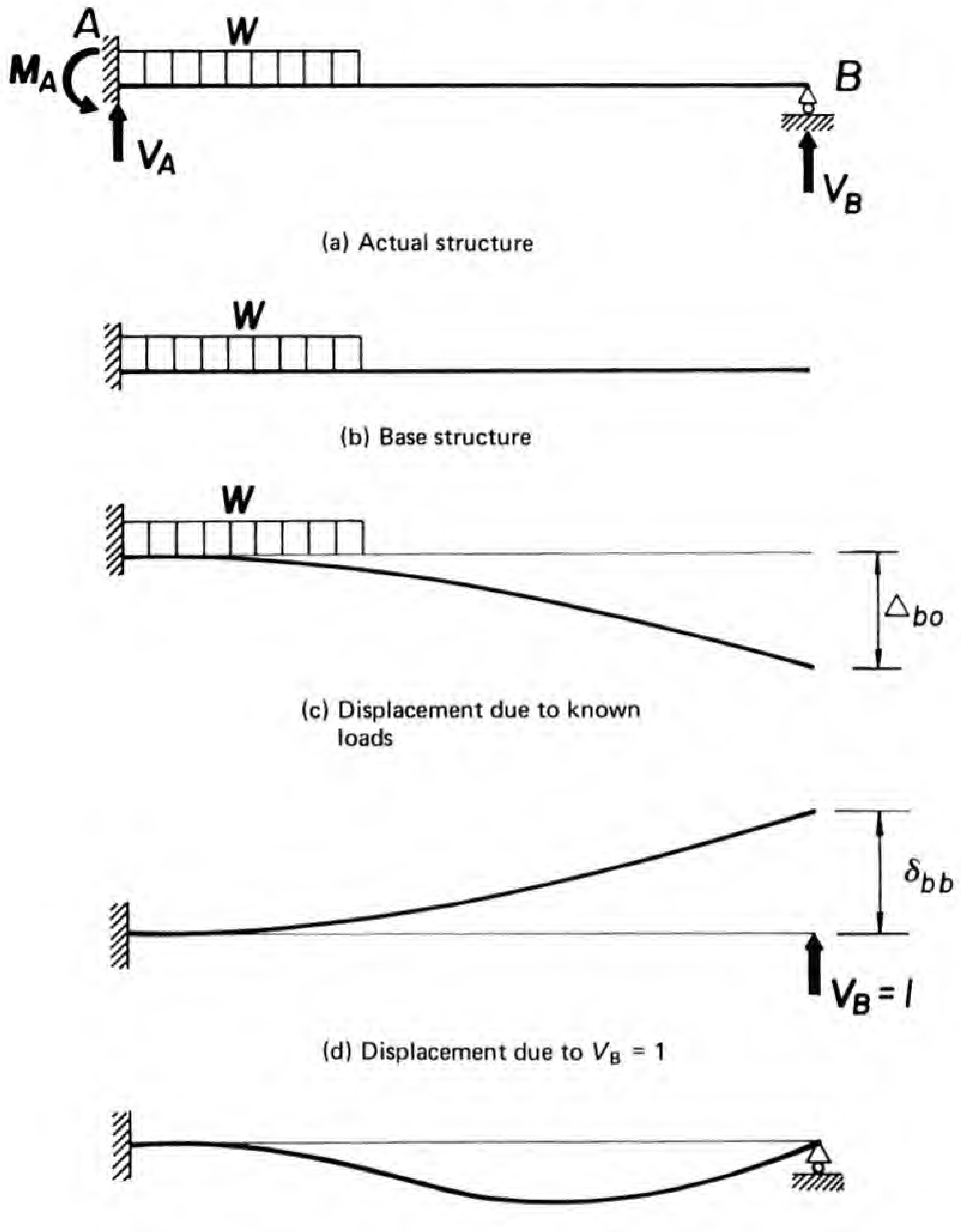
2.1 INTRODUCTION

For statically indeterminate structures there will be an indefinite number of combinations of redundant forces which will satisfy equilibrium conditions. However, among them there will be only one set of values that will simultaneously satisfy the requirements of equilibrium and compatibility. Compatibility places constraints on the displacements of a structure to ensure continuity and that the structure conforms to the displacement boundary conditions prescribed by the supports.

The methods of consistent displacement are based on the concept of equilibrium of forces and compatibility of displacements which may be stated as follows: Given a set of forces applied on a statically indeterminate structure, the reactions must assume such values that satisfy not only the conditions of static equilibrium with the applied loads but also the conditions of compatibility. The general method of consistent deformation is applicable for analysing all types of indeterminate structures. It is also applicable whether the structure is subjected to external loading, temperature changes, movements of supports, fabrication errors, or any other cause. Of course, there are other methods that are definitely superior for certain specific structures or loading conditions, but methods of consistent deformation are the most versatile and general.

2.2 ANALYSIS OF BEAMS

The principle of *consistent displacement* can best be illustrated by considering singly indeterminate structures. As a simple and classic example of this method consider a propped cantilever beam as shown in Fig. 2.1. The beam has three unknown reactions V_A, M_A and V_B and is therefore statically indeterminate to



(a) Actual structure

(b) Base structure

(c) Displacement due to known loads

(d) Displacement due to $V_B = 1$

(e)

Figure 2.1

the first degree. Any one of the unknown reactions may be taken as the redundant. A stable and determinate primary structure may be formed by determinate primary structure by selecting as the redundant the vertical reaction at the right support, V_B, as shown in Fig. 2.1(b).

The displacement of the cantilever beam AB may be considered to consist of the superposition of two independent displacements:

Δ_{bo} = upward deflection at B of the base structure due to the known applied loads only

Δ_{bb} = upward deflection at B of the base structure due to the redundant V_B

It may be noted that it is not possible to evaluate Δ_{bb} prior to the evaluation of V_B. However, by applying the principle of superposition such that $\Delta_{bb} = \delta_{bb}V_B$, where

δ_{bb} = upward deflection at B of the base structure due to a unit upward load at B

then, the condition that the support at B is rigid requires its displacement, Δ_B, the algebraic sum of displacements due to the applied loads and the redundant, must be zero. This geometric condition, defined as the *equation of consistent deformation*, may be writtten as

$$\Delta_B = \delta_{bo} + V_B\delta_{bb} = 0 \tag{2.1}$$

or

$$V_B = -\frac{\Delta_{bo}}{\delta_{bb}} = -\frac{\int \frac{Mmdx}{EI}}{\int \frac{m^2dx}{EI}} \tag{2.2}$$

where M is the moment in the base structure due to the applied loads and m is the moment due to a unit load acting at B.

It is noted that if V_B acts in the same direction as Δ_{bb}, a negative value is obtained which indicates that the assumed direction is wrong. Conversely, a positive value for V_B indicates that the assumed direction is correct. In general, it must be noted that the magnitude of the true reaction V_B is that required to restore the end B of the beam to its original position level with A.

In a similar manner, if M_A which is the moment reaction at A is taken as the redundant, the applied loads will cause the tangent at A to rotate through an angle θ_{ao}. If the rotation due to a unit moment at A is taken as θ_{aa} the moment M_A necessary to rotate the tangent at A to the original horizontal position is

$$M_A = \frac{\theta_{ao}}{\theta_{aa}} = \frac{\int \frac{Mm_\theta dx}{EI}}{\int \frac{m_\theta^2 dx}{EI}} \tag{2.3}$$

where m_θ is the moment due to a unit moment acting at A.

The analysis of beams of higher degree of indeterminacy follows closely the procedure described above. For a beam with n degrees of indeterminacy, n redundants are selected which will be removed from the structure and replaced by n effectively equivalent redundant forces $X_1, X_2, \ldots X_n$. All these redundant

forces and the given external loads are applied on the base structure such that their magnitudes must cause the displacements at the points of application of the n redundants of the base structure to be equal to the displacement of the corresponding points on the actual structure.

Consider the four-span continuous beam of Fig. 2.2. The beam has three redundant reactions which can be chosen in a variety of ways, one of which is shown in Fig. 2.2(a).

At this stage it is convenient to follow a definite notation for the various redundant forces and displacements. The redundant forces X_a, X_b and X_c are recognised and identified by single subscripts which denote their point of application. The displacements are identified by double subscripts: the first subscript denotes the point on the base structure at which the displacement occurs, and the second subscript is used to denote the force producing the displacement. If, for example, the points A, B, C, etc. are the points on the base structure where the redundants occur, then,

X_a = the redundant force at point A

Δ_{ao} = the displacement in the base structure at point A in the direction of X_a, caused by the actual applied loads acting on the structure

δ_{aa} = displacement in the direction of X_a in the base structure caused by $X_a = 1$ and no other load acting

δ_{ab} = displacement in the base structure at A in the direction of X_a caused by $X_b = 1$ acting alone

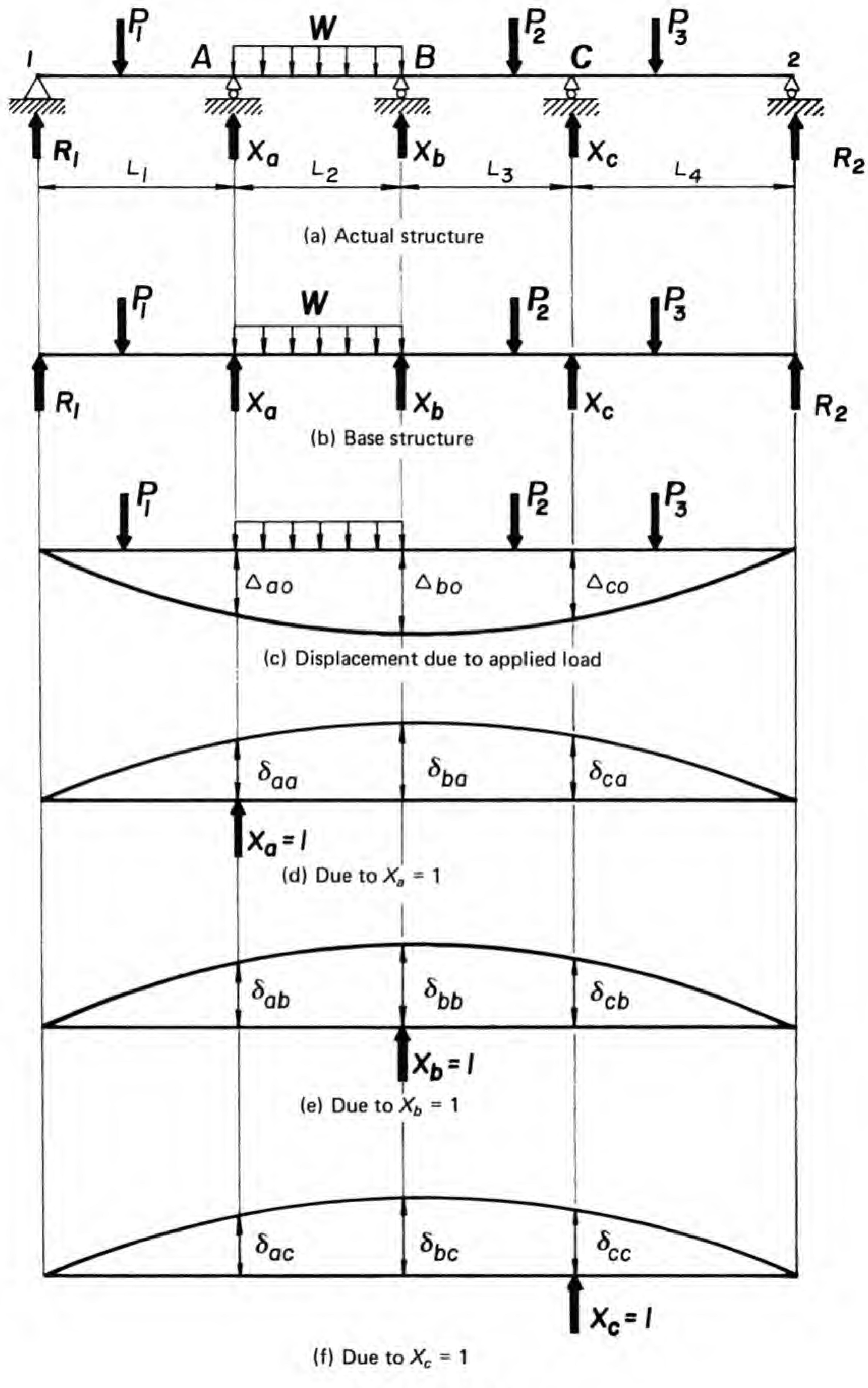
δ_{ac} = displacement in the base structure at A in the direction of X_a caused by $X_c = 1$ acting alone

Since the displacements at A, B, and C should be zero, the reactions X_a, X_b and X_c must have values such that compatibility condition is satisfied. Thus, using the above notation in the superposition equations, which gives as many equations as there are redundants, the equations may be written as follows:

$$\begin{aligned}
&\Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} = 0\\
&\Delta_{bo} + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} = 0\\
&\Delta_{co} + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} = 0
\end{aligned} \qquad [2.4]$$

Since $\delta_{ab} = \delta_{ba}$, $\delta_{ac} = \delta_{ca}$, etc. by Maxwell's principle of reciprocal deflections, [2.4] may be written as

$$\begin{aligned}
&\Delta_{ao} + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} = 0\\
&\Delta_{bo} + X_a\delta_{ab} + X_b\delta_{bb} + X_c\delta_{bc} = 0\\
&\Delta_{co} + X_a\delta_{ac} + X_b\delta_{bc} + X_c\delta_{cc} = 0
\end{aligned} \qquad [2.5]$$



P1
W
P2
P3
1
A
B
C
2
R1
Xa
Xb
Xc
R2
L1
L2
L3
L4
(a) Actual structure
(b) Base structure
Δao
Δbo
Δco
(c) Displacement due to applied load
δaa
δba
δca
Xa = 1
(d) Due to Xa = 1
δab
δbb
δcb
Xb = 1
(e) Due to Xb = 1
δac
δbc
δcc
Xc = 1
(f) Due to Xc = 1

Figure 2.2

The equations can be written in the following matrix form:

$$\begin{bmatrix} \Delta_{ao} \\ \Delta_{bo} \\ \Delta_{co} \end{bmatrix} + \begin{bmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ab} & \delta_{bb} & \delta_{bc} \\ \delta_{ac} & \delta_{bc} & \delta_{cc} \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [2.6]$$

EXAMPLE 2.1 *Determine the reactions and support moment of the continuous beam shown in Fig. 2.3.*

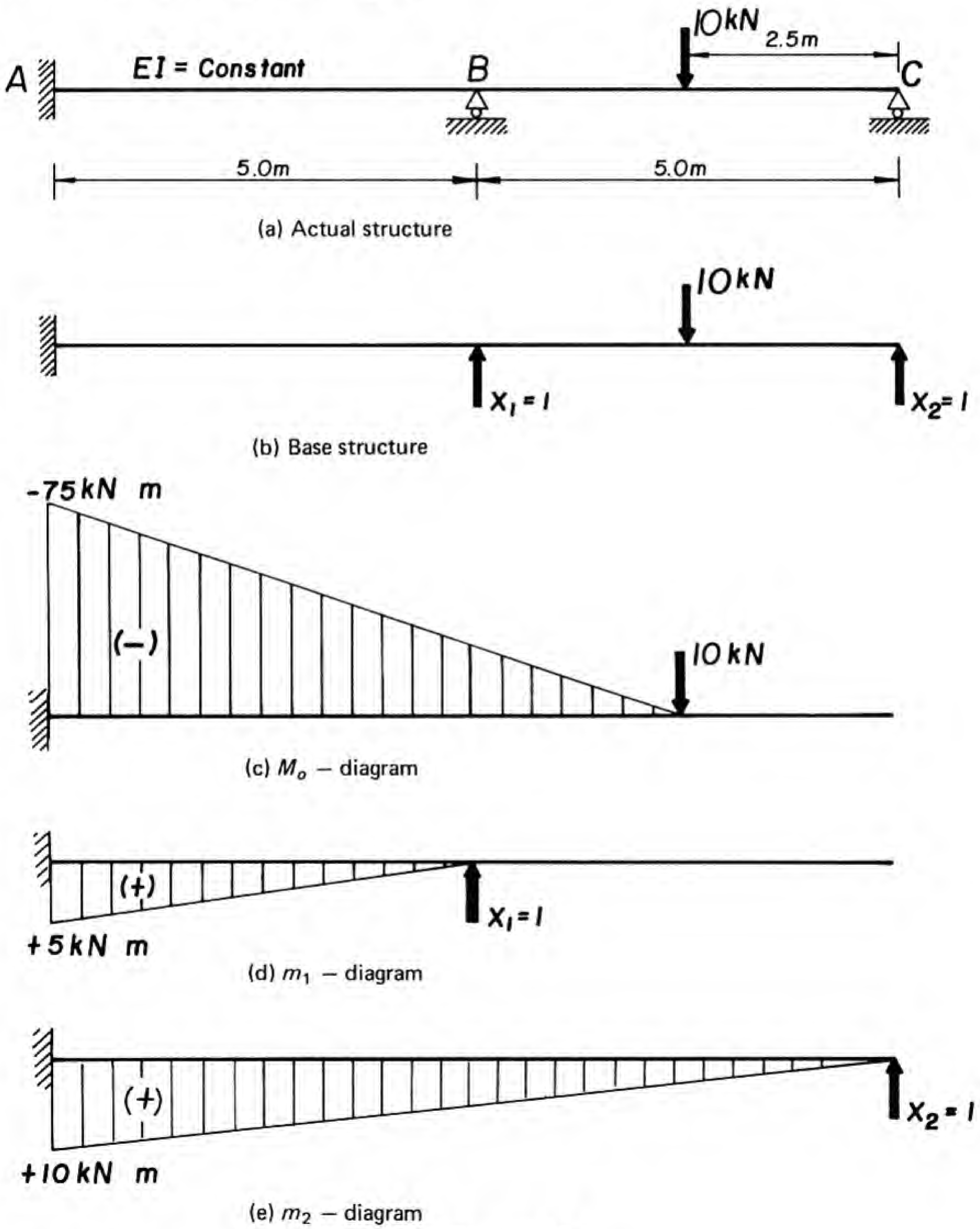


Figure 2.3

The beam is indeterminate to the second degree, and the redundants chosen are the reactions at B and C. The moment diagrams due to the applied load $X_1 = 1$ and $X_2 = 1$ are shown in Fig. 2.3(d) and (e) respectively.

The elastic equations are

$$\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} = 0$$

$$\Delta_{20} + X_1\delta_{12} + X_2\delta_{22} = 0$$

The displacements are obtained by graphic multiplication method:

$$EI\Delta_{10} = \left(-\frac{5 \times 5}{2}\right)\left(\frac{17.5}{3} \times \frac{7.5}{7.5}\right) = -729.17$$

$$EI\delta_{11} = \left(\frac{5 \times 5}{2}\right)\left(\frac{2 \times 5}{3}\right) = 41.67$$

$$EI\delta_{12} = \left(\frac{5 \times 5}{2}\right)\left(\frac{25}{3}\right) = 104.17$$

$$EI\Delta_{20} = \left(-\frac{75 \times 7.5}{2}\right)\left(\frac{22.5}{3}\right) = -2109.38$$

$$EI\delta_{22} = \left(\frac{10 \times 10}{2}\right)\left(\frac{2 \times 10}{3}\right) = 333.33$$

Substituting the δ terms into the elastic equations:

$$-729.17 + 41.67X_1 + 104.17X_2 = 0$$

$$-2109.38 + 104.17X_1 + 333.33X_2 = 0$$

The solution of the simultaneous equations is

$$X_1 = 7.68 \text{ kN}$$

$$X_2 = 3.93 \text{ kN}$$

Thus,

$$R_B = 7.68 \text{ kN (upward)}$$

$$R_C = 3.93 \text{ kN (upward)}$$

From statics

$$R_A = 1.61 \text{ kN (downward)}$$

$$M_A = 2.70 \text{ kN (clockwise)}$$

Alternative Solution

The moment reaction and the vertical reaction at A are chosen as redundants. The moment diagram due to the applied loads $X_1 = 1$ and $X_2 = 1$ are shown in Fig. 2.4(d) and (e) respectively.

The displacements are

$$EI\Delta_{10} = \left(\frac{2.5 \times 2.5}{2}\right)\left(\frac{2}{3} \times 12.5\right) + \left(\frac{12.5 \times 2.5}{2}\right)\left(2.5 \times \frac{1}{3} \times 2.5\right)$$

$$= 78.13$$

$$EI\delta_{11} = 2\left(\frac{5.0 \times 5.0}{2}\right)\left(\frac{2}{3} \times 5\right)$$

$$= 83.33$$

$$EI\Delta_{12} = \left(-\frac{5.0 \times 5.0}{2}\right)(1.0) - \left(\frac{5.0 \times 5.0}{2}\right)\left(\frac{2 \times 1.0}{3}\right)$$

$$= -20.83$$

$$EI\Delta_{20} = -\left(\frac{12.5 \times 2.5}{2}\right)\left(\frac{2}{3} \times \frac{2.5}{5}\right) - \left(\frac{12.5 \times 2.5}{2}\right)\left(2.5 + \frac{2.5}{3}\right)\frac{1}{5}$$

$$= -15.63$$

$$EI\delta_{22} = (1.0 \times 5.0)(1.0) + \left(\frac{1.0 \times 5.0}{2}\right)\left(\frac{2}{3}\right)$$

$$= 6.67$$

Substituting into the elastic equations:

$$78.13 + 83.33X_1 - 20.83X_2 = 0$$

$$-15.63 - 20.83X_1 + 6.67X_2 = 0$$

The solution of the simultaneous equations is

$$X_1 = -1.61 \text{ kN}$$

$$X_2 = -2.70 \text{ kN m}$$

The reactions are

$R_A = 1.61$ kN (upward)

$M_A = 2.70$ kN m (clockwise)

$R_B = 7.68$ kN (upward)

$R_C = 3.93$ kN (upward)

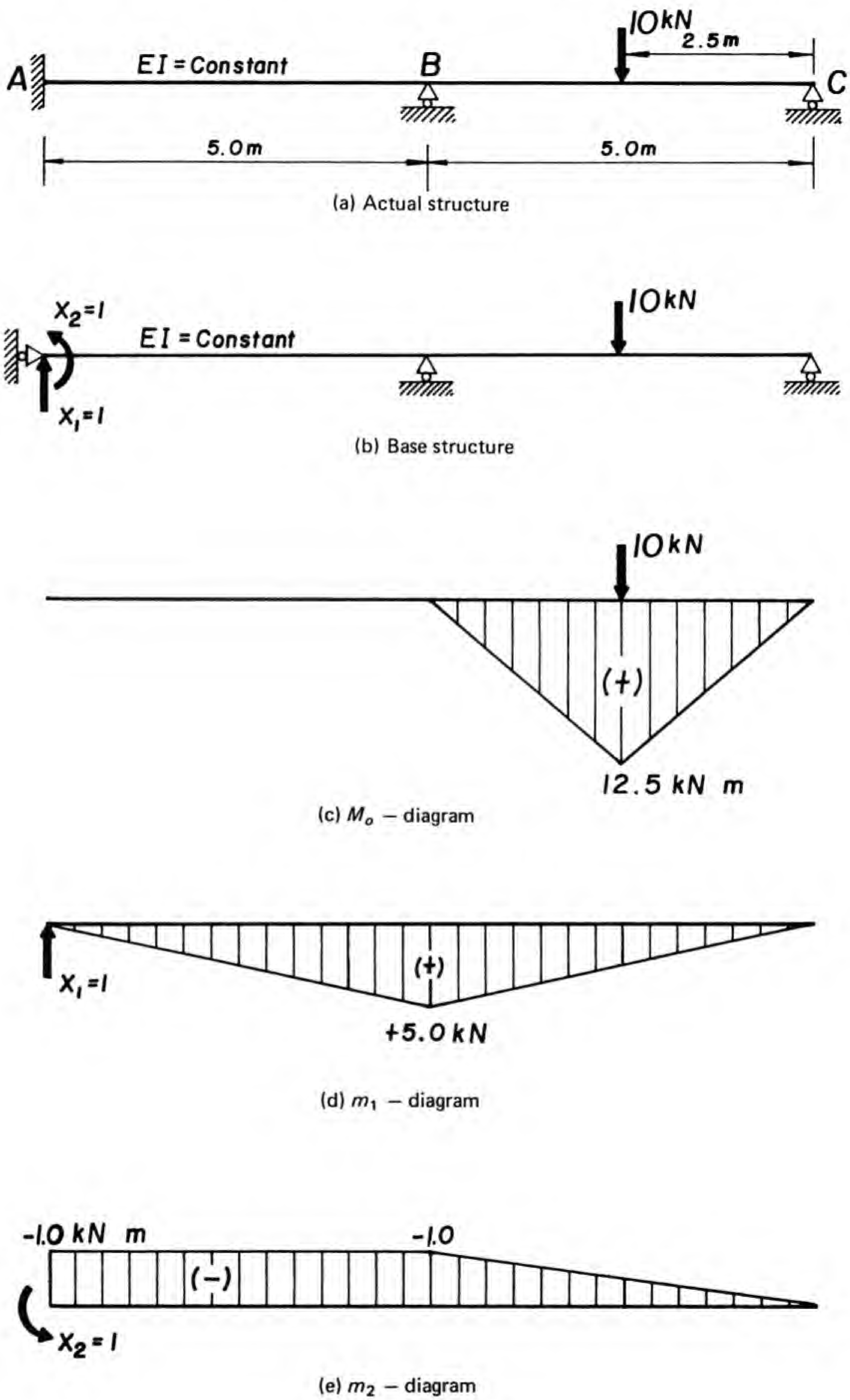

10kN
2.5m
A
EI = Constant
B
C
5.0m
5.0m
(a) Actual structure
X₂=1
10kN
EI = Constant
X₁=1
(b) Base structure
10kN
(+)
12.5 kN m
(c) M₀ – diagram
(+)
X₁=1
+5.0 kN
(d) m₁ – diagram
-1.0 kN m
-1.0
(−)
X₂ = 1
(e) m₂ – diagram

Figure 2.4

2.3 ANALYSIS OF TRUSSES

A statically indeterminate truss with external redundant reaction or internal redundant member may be analysed by a procedure closely analogous to that followed in beams. The analysis of trusses with a redundant reaction consists of choosing a base structure by *removing* the redundant reactions. Acting on this base structure are the applied loading and the redundant reactions. Then the condition of compatibility is applied such that the displacements in the direction of the redundants become zero. In a similar manner, when the truss has redundant members, the base structure is obtained by *cutting* the redundant members and replacing it by a pair of forces and then applying the condition of compatibility. Take, for example, the truss shown in Fig. 2.5. The truss is internally indeterminate to the first degree.

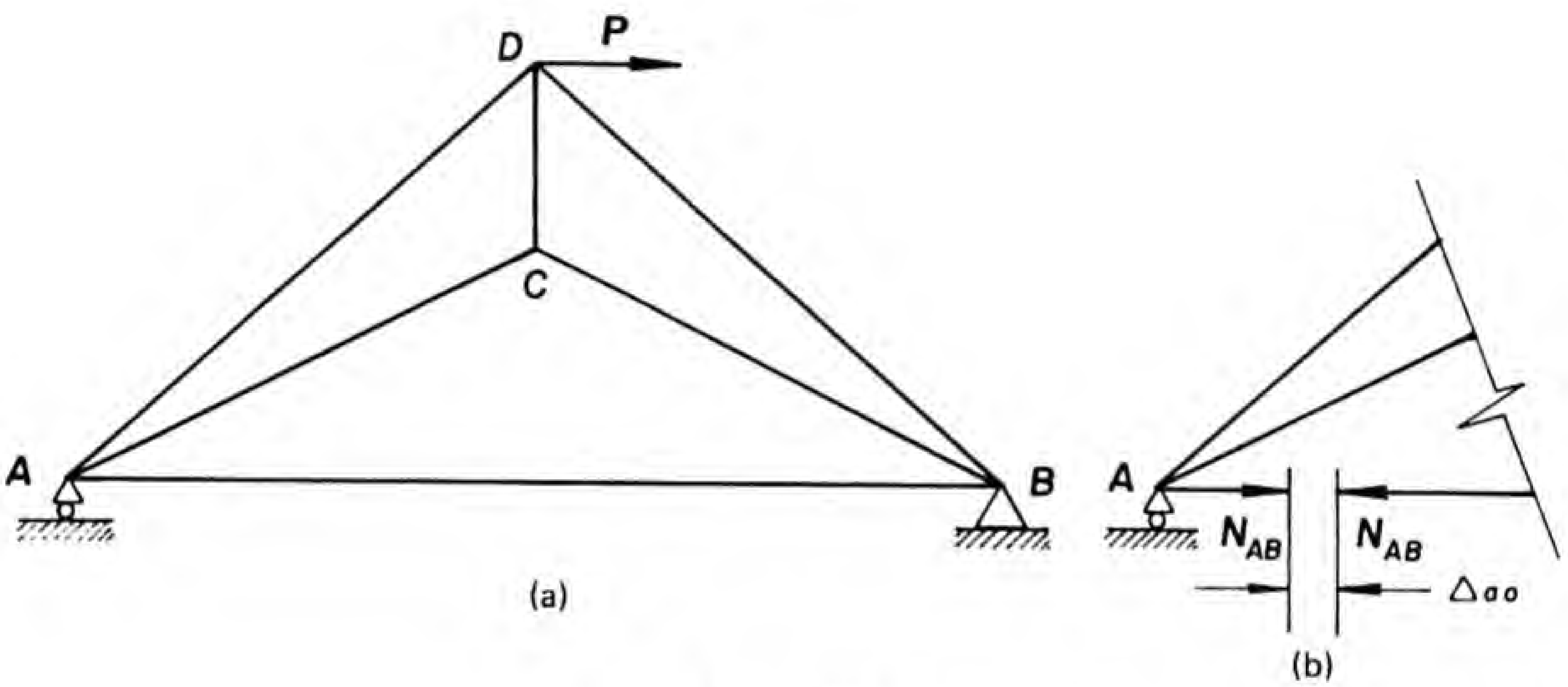


Figure 2.5

In this truss, any member may be considered redundant. Choosing member AB as the redundant, the redundant member is *removed* by cutting it at any section. Due to the effect of the external load P on the base structure, the two faces of the cut member AB will be displaced by Δ_{ao}. Now, applying a pair of forces N_{AB} as shown in Fig. 2.5(b), such that the relative displacement of the actual truss at the cut surface is zero, gives the following relationship:

$$\Delta_{ao} + N_{AB}\delta_{aa} = 0 \qquad [2.7]$$

where δ_{aa} is the relative displacement of the cut faces due to $N_{AB} = 1$. The internal force in the redundant member is

$$N_{AB} = -\frac{\Delta_{ao}}{\delta_{aa}} \qquad [2.8]$$

But from virtual work principle,

$$\Delta_{ao} = \Sigma \frac{NnL}{EA}$$
$$\delta_{aa} = \Sigma \frac{n^2 L}{EA} \qquad [2.9]$$

where N = force in any member due to the external applied load acting on the base structure

n = force in any member due to a unit pair of forces applied at the cut faces of the member.

Thus

$$N_{\mathrm{AB}} = \frac{\Sigma \frac{NnL}{EA}}{\Sigma \frac{n^2 L}{EA}} \qquad [2.10]$$

Note that the summation in the denominator is taken over the whole truss, and the summation in the numerator applies only on the base structure.

The analysis of trusses of higher degree of indeterminacy follows closely the procedure described above. Consider, for example, the truss shown in Fig. 2.6 which is externally statically indeterminate to the second degree. If the supports at B and C are removed, a simple truss supported at A and D will be the basic determinate truss. The deflected bottom chord due to the applied loading is shown in Fig. 2.6(b). The displacements at B and C are determined from the expressions

$$\Delta_{bo} = \Sigma \frac{Nn_b L}{EA}$$
$$\Delta_{co} = \Sigma \frac{Nn_c L}{EA} \qquad [2.11]$$

Figure 2.6(b) shows the displacements at B and C due to a unit load applied at B, and in Fig. 2.6(c) due to a unit load applied at C. The vertical displacements are determined from the expressions

$$\delta_{bb} = \Sigma \frac{n_b^2 L}{EA}$$

$$\delta_{bc} = \delta_{cb} = \Sigma \frac{n_b n_c L}{EA} \qquad [2.12]$$

$$\delta_{cc} = \Sigma \frac{n_c^2 L}{EA}$$

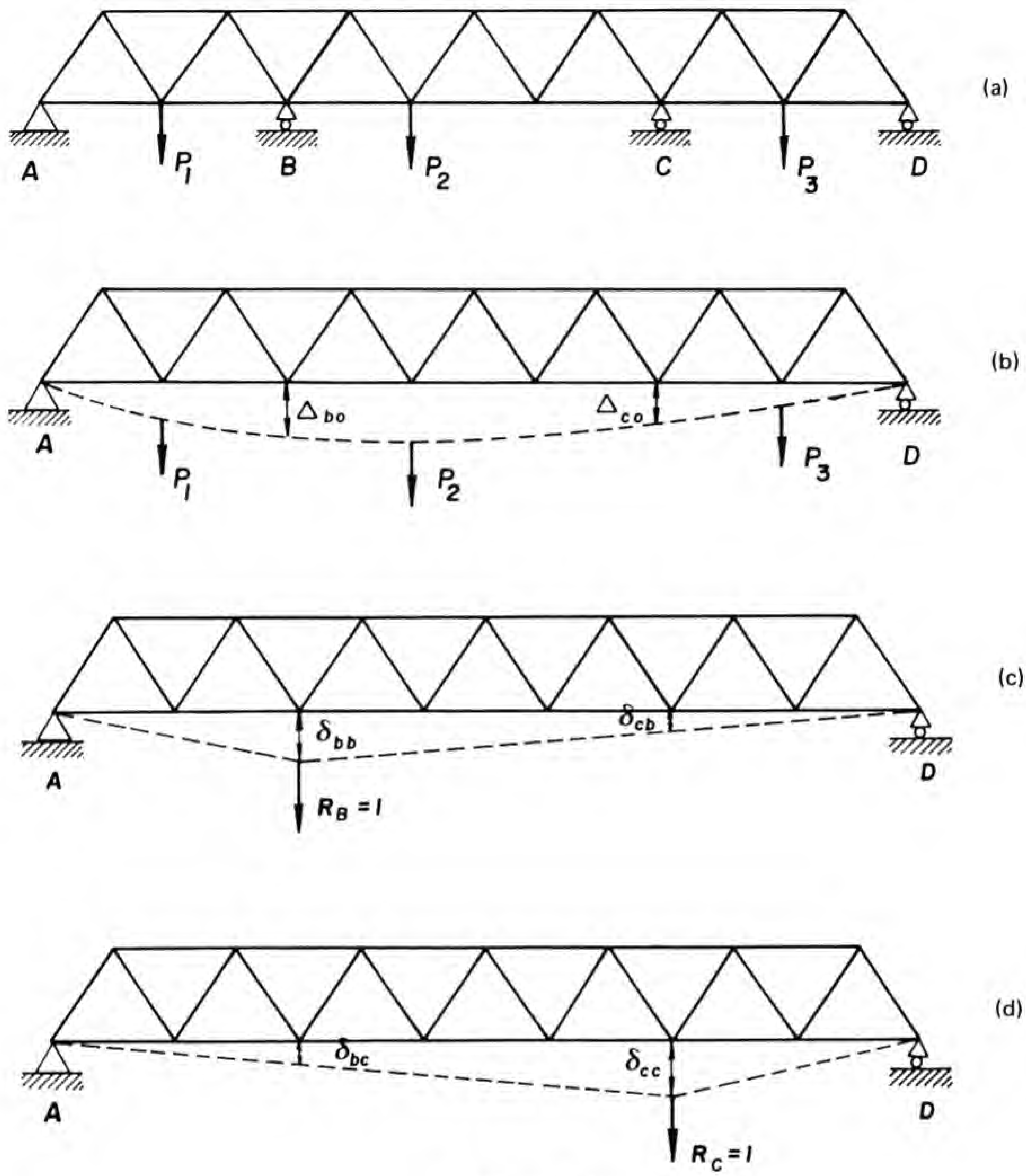


Figure 2.6

In the above expressions N stands for forces in the members due to the external applied loads on the base structure, and N_b and N_c are the forces in the members due to a unit load applied at B and C, respectively.

The conditions of compatibility required from which R_B and R_C can be determined are

$$\begin{aligned} \Delta_{bo} + R_b\delta_{bb} + R_c\delta_{bc} &= 0 \\ \Delta_{co} + R_b\delta_{bc} + R_c\delta_{cc} &= 0 \end{aligned} \qquad [2.13]$$

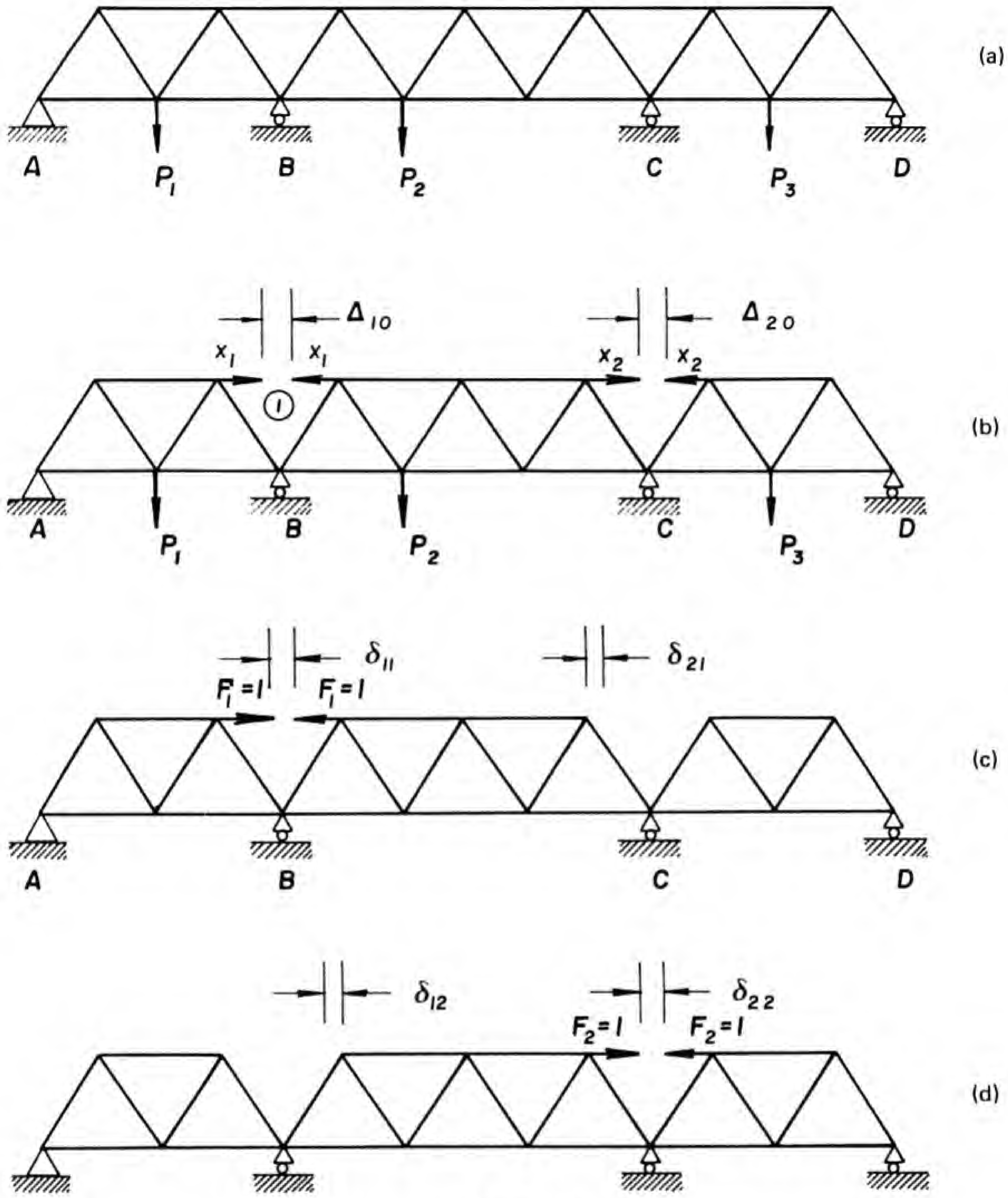


Figure 2.7

Consider again the truss shown in Fig. 2.6. If the redundants are taken as the bar forces X_1 and X_2 shown in Fig. 2.7(b), then the determinate truss is three independent simple span trusses. Due to the effect of the external loading on the base structure, the two faces of the cut members 1 and 2 will be displaced by Δ_{10} and Δ_{20}, respectively. After applying a pair of forces $F_1 = 1$ and $F_2 = 1$ as shown in Fig. 2.7(c) and (d), the corresponding relative displacements of the cut

faces can be determined. The displacements are

$$\begin{aligned}
\Delta_{10} &= \Sigma \frac{Nn_1L}{EA} \\
\Delta_{20} &= \Sigma \frac{Nn_2L}{EA} \\
\delta_{11} &= \Sigma \frac{n_1^2L}{EA} \\
\delta_{12} &= \delta_{21} = \Sigma \frac{n_1n_2L}{EA} \\
\delta_{22} &= \Sigma \frac{n_2^2L}{EA}
\end{aligned} \qquad [2.14]$$

The conditions of compatibility from which X_1 and X_2 can be determined are

$$\begin{aligned}
\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} &= 0 \\
\Delta_{20} + X_1\delta_{12} + X_2\delta_{22} &= 0
\end{aligned} \qquad [2.15]$$

EXAMPLE 2.2 *Find the reaction at B and the bar force in member BF, of the truss in Fig. 2.8. The cross-sectional area of the members in cm² are shown in parentheses. E is constant.*

The given truss is indeterminate to the second degree; it has one redundant member (internal indeterminacy) and one redundant reaction (external indeterminacy).

A base structure is obtained by removing the reaction at B and cutting the diagonal member BF. The two conditions of compatibility are:

$$\Delta_B + R_B\delta_{bb} + F_{BF}\delta_{bf} = 0$$

$$\Delta_F + R_B\delta_{bf} + F_{BF}\delta_{ff} = 0$$

The displacements are computed in tabular form as shown in Table 2.1. Substituting the displacements:

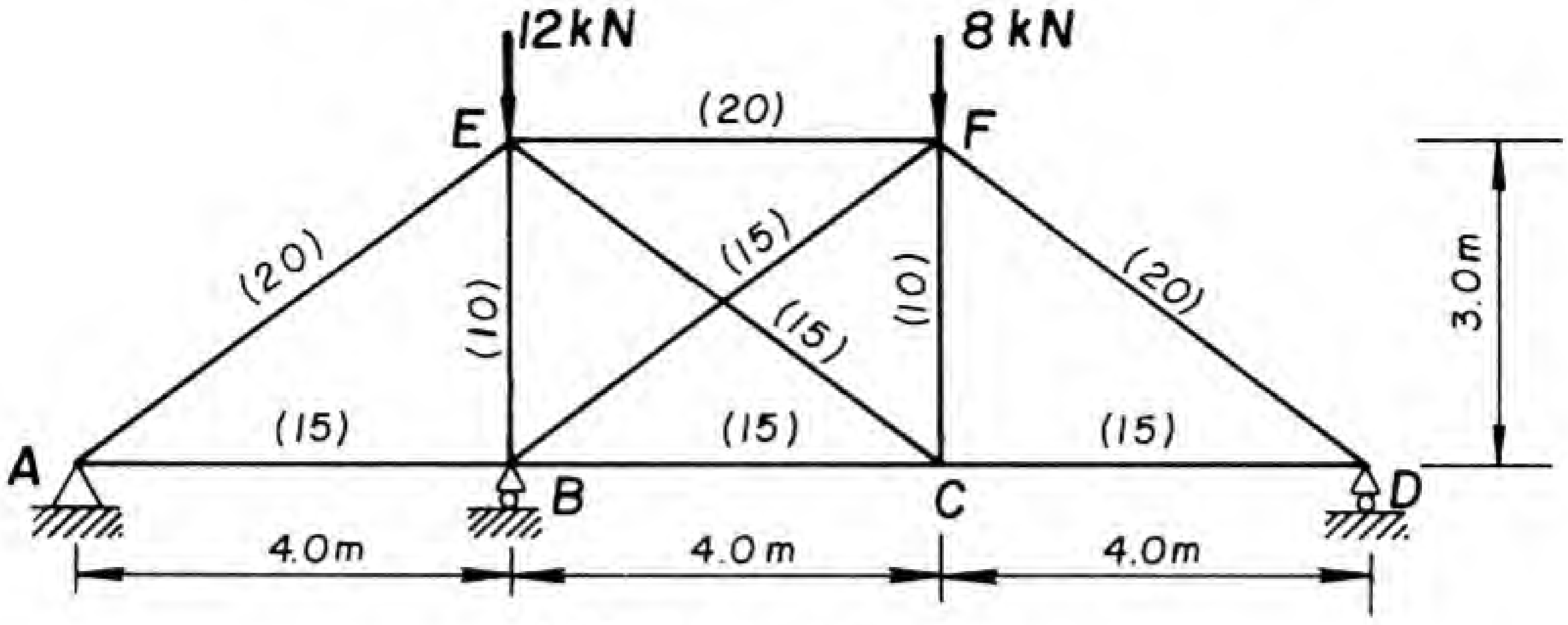
$$-1680.8 + 134.91R_B + 54.52F_{BF} = 0$$

$$-202.3 + 54.52R_B + 118.1F_{BF} = 0$$

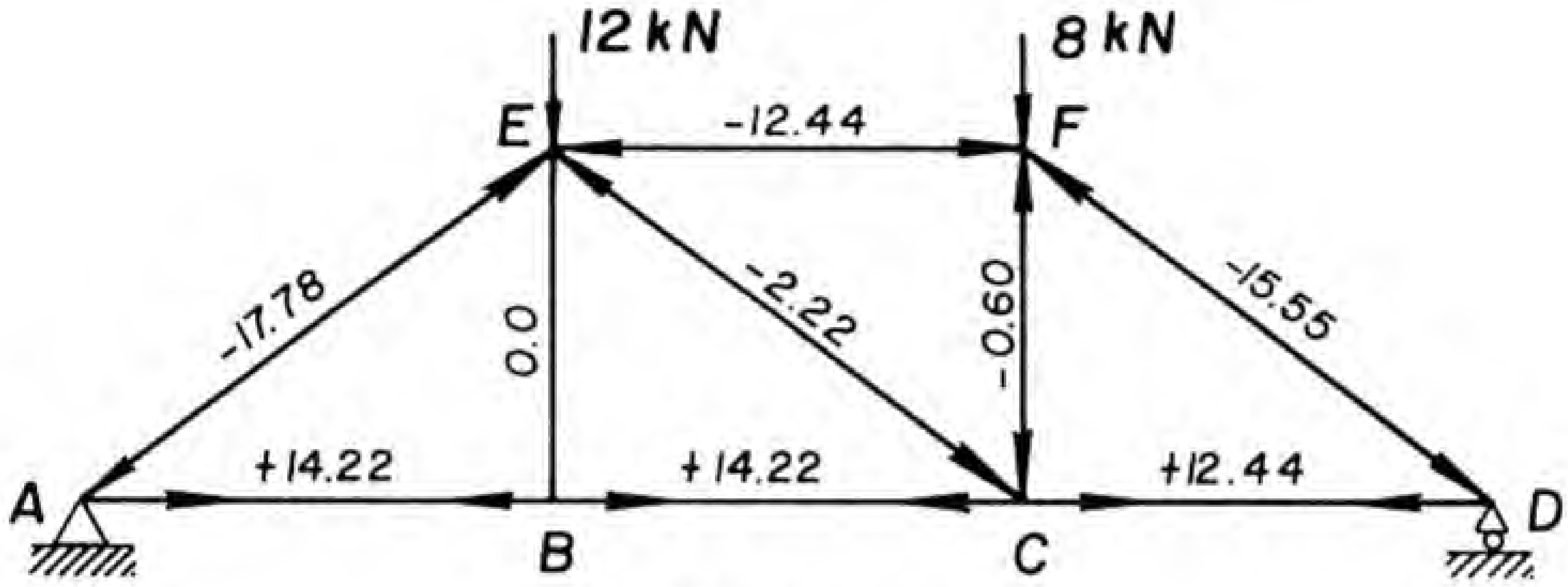
The solution of the simultaneous equations is

$$R_B = 14.44 \text{ tonnes (upward)}$$

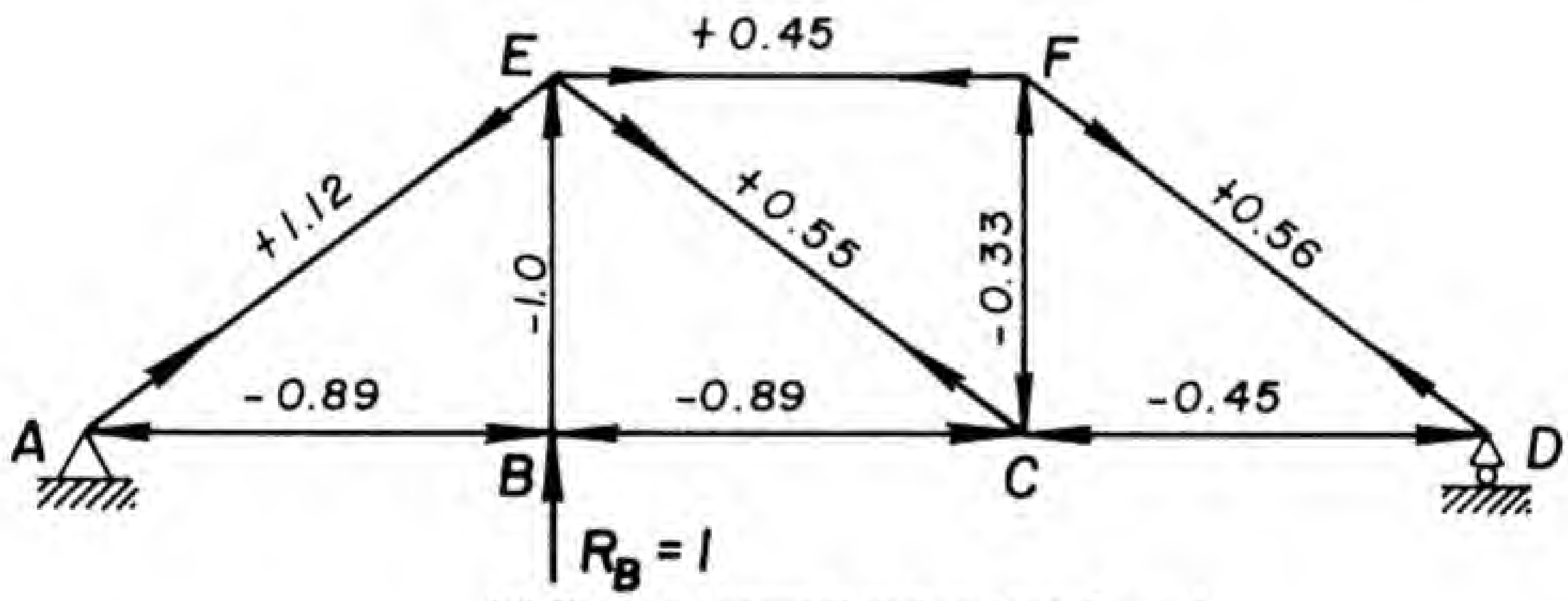
$$F_{BF} = -4.95 \text{ tonnes (compression)}$$



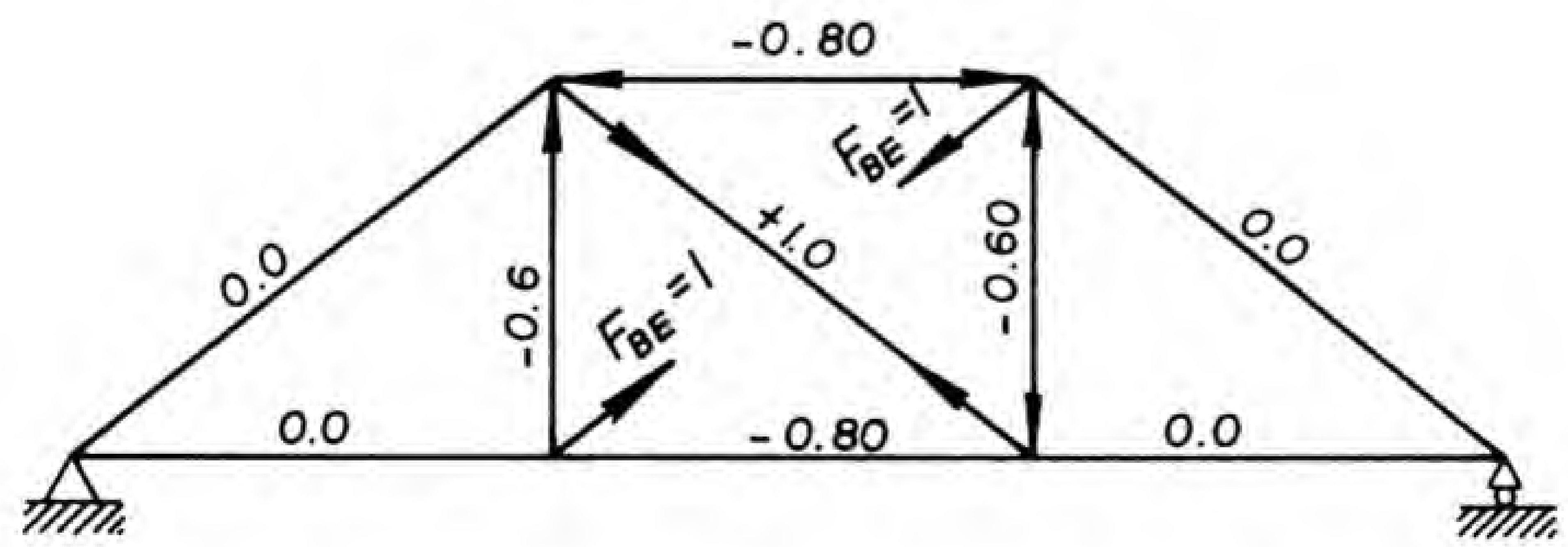
(a) Actual structure



(b) Base structure



(c) Due to external redundant $R_B = 1$



(d) Due to internal redundant $F_{BE} = 1$

Figure 2.8

Table 2.1

Member	L cm	A cm^2	$\frac{L}{A}$	N	n_B $(R_B = 1)$	n_{BF} $F_{BF} = 1$	$\frac{Nn_B L}{A}$	$\frac{Nn_{BF} L}{A}$	$\frac{n_B^2 L}{A}$	$\frac{n_{BF}^2 L}{A}$	$\frac{n_B n_{BF} L}{A}$
AE	500	20	25	17.78	+1.12	0	−497.84	0	+31.36	0	0
EF	400	20	20	−12.44	+0.45	−0.80	−111.96	+199.04	+ 4.05	+ 12.80	− 7.20
DF	500	20	25	−15.55	+0.55	0	−213.81	0	+ 7.56	0	0
BE	300	10	30	0	−1.00	−0.60	0	0	+30.0	+ 10.80	+18.0
CE	500	15	33.33	+ 2.22	+0.56	+1.00	− 41.44	− 73.99	+10.45	+ 33.30	+18.66
BF	500	15	33.33	0	0	1.00	0	0	0	+ 33.30	0
CF	300	10	30	− 1.33	−0.33	−0.60	+ 13.17	− 23.94	+ 3.27	+ 10.80	+ 5.94
AB	400	15	26.67	+14.22	−0.90	0	−339.81	0	+21.41	0	0
BC	400	15	26.67	+14.22	−0.90	−0.80	−339.81	−303.40	+21.41	+ 17.10	+19.12
CD	400	15	26.67	−12.44	−0.45	0	−149.3	0	+ 5.40	0	0
						Σ	−1680.8	−202.29	134.91	+118.1	54.52

2.4 ANALYSIS OF FRAMES

A framed structure is composed of an interconnected assemblage of beams and columns. A frame is said to be *rigid* if the members are rigidly connected. The basic analysis of statically indeterminate frames by the method of consistent deformation is essentially an extension of the same principle encountered in dealing with beams.

The members in frames are usually subjected to both axial and bending stresses; however, the axial stresses in the members of rigid frames are in most cases small compared with that of bending stresses. Thus, in computing the displacements in rigid frames for the conditions of consistent deformation, the effects of the axial stresses are usually neglected and the effects of bending stresses only are considered. This, however, does not mean that there are no axial forces in the members even if the change in the length of the members of rigid frames has insignificant effect on the values of the redundants.

To formulate the equations for the general case of multiply redundant structures, consider the frame shown in Fig. 2.9, which is triply statically indeterminate. Let the three support reactions at A be chosen as the redundants. When these redundants are removed, A will be displaced vertically and horizontally and will also rotate.

It will be seen that it will be convenient to adopt a slightly different notation with numerical subscripts for the redundants and displacements, which are defined as

$\Delta_{10}, \Delta_{20}, \Delta_{30}$ = displacements at A in the directions of X_1, X_2 and X_3 respectively, due to the applied loads on the base structure

$\delta_{11}, \delta_{21}, \delta_{31}$ = displacements at A on the base structure in the directions of X_1, X_2 and X_3 respectively, due to $X_1 = 1$ acting alone

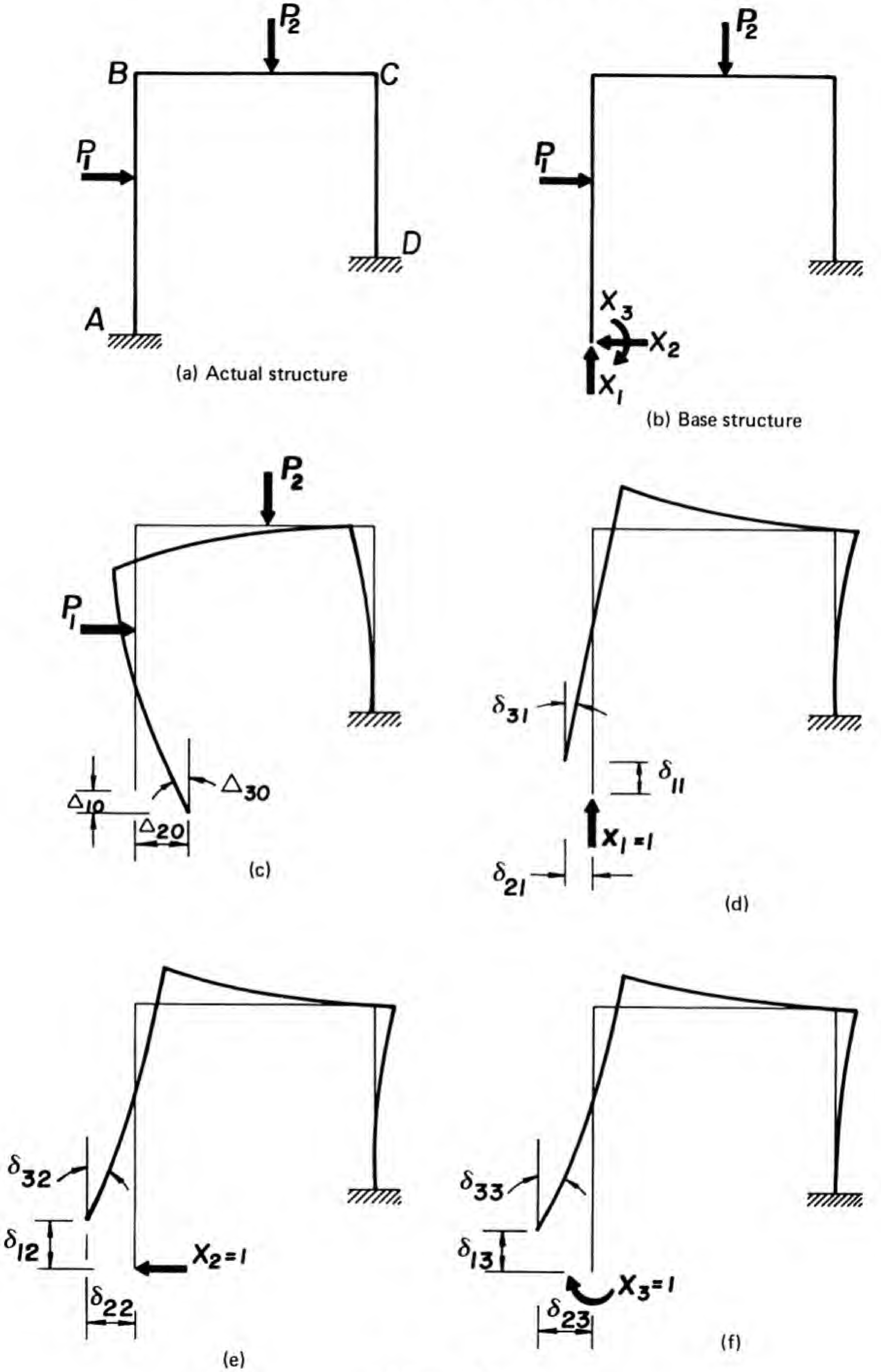
$\delta_{12}, \delta_{22}, \delta_{32}$ = the above displacements on the base structure due to $X_2 = 1$ acting alone

$\delta_{13}, \delta_{23}, \delta_{33}$ = the above displacements on the base structure due to $X_3 = 1$ acting alone

If it is known that there are no support displacements, the equations of consistent deformation are

$$
\begin{aligned}
&\Delta_{10} + X_1\delta_{11} + X_2\delta_{12} + X_3\delta_{13} = 0 \\
&\Delta_{20} + X_2\delta_{12} + X_2\delta_{22} + X_3\delta_{23} = 0 \qquad [2.16] \\
&\Delta_{20} + X_3\delta_{13} + X_2\delta_{23} + X_3\delta_{33} = 0
\end{aligned}
$$

The general equation for a structure with n redundants may then be written in



P2
B
C
P1
D
A
(a) Actual structure
P2
P1
X3
X2
X1
(b) Base structure
P2
P1
Δ10
Δ20
Δ30
(c)
δ31
δ11
X1 = 1
δ21
(d)
δ32
δ12
X2 = 1
δ22
(e)
δ33
δ13
X3 = 1
δ23
(f)

Figure 2.9

matrix form as

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \dots \\ \Delta_{n0} \end{bmatrix} + \begin{bmatrix} \delta_{11} + \delta_{12} + \dots + \delta_{1n} \\ \delta_{12} + \delta_{22} + \dots + \delta_{2n} \\ \dots \quad \dots \\ \delta_{1n} + \delta_{2n} + \dots + \delta_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \qquad [2.17]$$

These equations, sometimes referred to as the elastic equations, form the basis for several different methods of analysing statically indeterminate structures. The coefficients $\delta_{11}, \delta_{12} \dots$ of the redundants on the base structures, which are the displacements due to unit loads are known as *flexibility coefficients* or *influence coefficients*.

Whenever support displacements occur, the right-hand side of the equations may be suitably adjusted before solving the simultaneous equations.

In the general case where deflections occur as a consequence of flexural and axial deformation of members of the structure, displacements in the base structure due to the applied loads may be written in the form

$$\Delta_{10} = \int \frac{Mmdx}{EI} + \sum \frac{NnL}{EA} \qquad [2.18]$$

and the flexibility coefficients are

$$\delta_{ij} = \int \frac{m_i m_j dx}{EI} + \sum \frac{n_i n_j L}{EA} \qquad [2.19]$$

EXAMPLE 2.3 *Determine the reaction components at support D of the frame of Fig. 2.10.*

Since the horizontal and vertical displacements and the rotations at support D must be zero, the compatibility equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The displacement coefficients are evaluated using the graphic multiplication method:

$$EI\Delta_{10} = \left(-\frac{3.0 \times 1.5}{2}\right)(2.5) + (-5.0 \times 6.0)(3) + \left(-\frac{3 \times 9}{2}\right)(3.0)$$

$$= -136.125$$

$$EI\Delta_{20} = \left(-\frac{3.0 \times 1.5}{2}\right)(-3) + (0.5 \times 6)(+0.5) + \frac{(3 \times 9)}{2}(1.0)$$

$$= +8.25$$

$$EI\Delta_{30} = -\left(\frac{3.0 \times 1.5}{2}\right)(-1.0) + (-5 \times 6)(-1.0) + \left(\frac{3 \times 9}{2}\right)(-1.0)$$

$$= 45.75$$

$$EI\delta_{11} = \left(\frac{3 \times 1.5}{2}\right)(2.0) + (3.0)(5.0)(3.0)$$

$$= 49.5$$

$$EI\delta_{12} = \left(\frac{1.5 \times 3}{2}\right)(-3.0) + (1.5)(-0.5)$$

$$= -14.25 = EI\delta_{21}$$

$$EI\delta_{13} = \left(\frac{1.5 \times 3}{2}\right)(-1.0) + (1.5 \times 5)(-1.0)$$

$$= -17.25 = EI\delta_{31}$$

$$EI\delta_{22} = \left(-\frac{3 \times 3}{2}\right)(-2.0) + \left(-\frac{1.5 \times 3}{2}\right)(-3.0) + \left(-\frac{3 \times 3}{2}\right)(-2.0)$$

$$+ \left(\frac{2 \times 2}{2}\right) \frac{4}{5}$$

$$= 34.17$$

$$EI\delta_{23} = \left(-\frac{3 \times 3}{2}\right)(-1.0) + \left(-\frac{1.5 \times 3}{2}\right)(-1.0) + \left(-\frac{3 \times 3}{2}\right)(-1.0)$$

$$+ \left(\frac{2 \times 2}{2}\right)(-1.0)$$

$$= 11.5 = EI\delta_{32}$$

$$EI\delta_{33} = (-1.0 \times 3)(-1.0) + (-1.0 \times 3)(-1.0) + (-1.0 \times 5)(-1.0)$$

$$= 9.5$$

Substituting these values into the elastic equations:

$$\begin{bmatrix} -136.125 \\ 8.25 \\ 45.75 \end{bmatrix} + \begin{bmatrix} 49.5 & -14.25 & -17.25 \\ -14.25 & 34.17 & 11.5 \\ -17.25 & 11.5 & 9.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

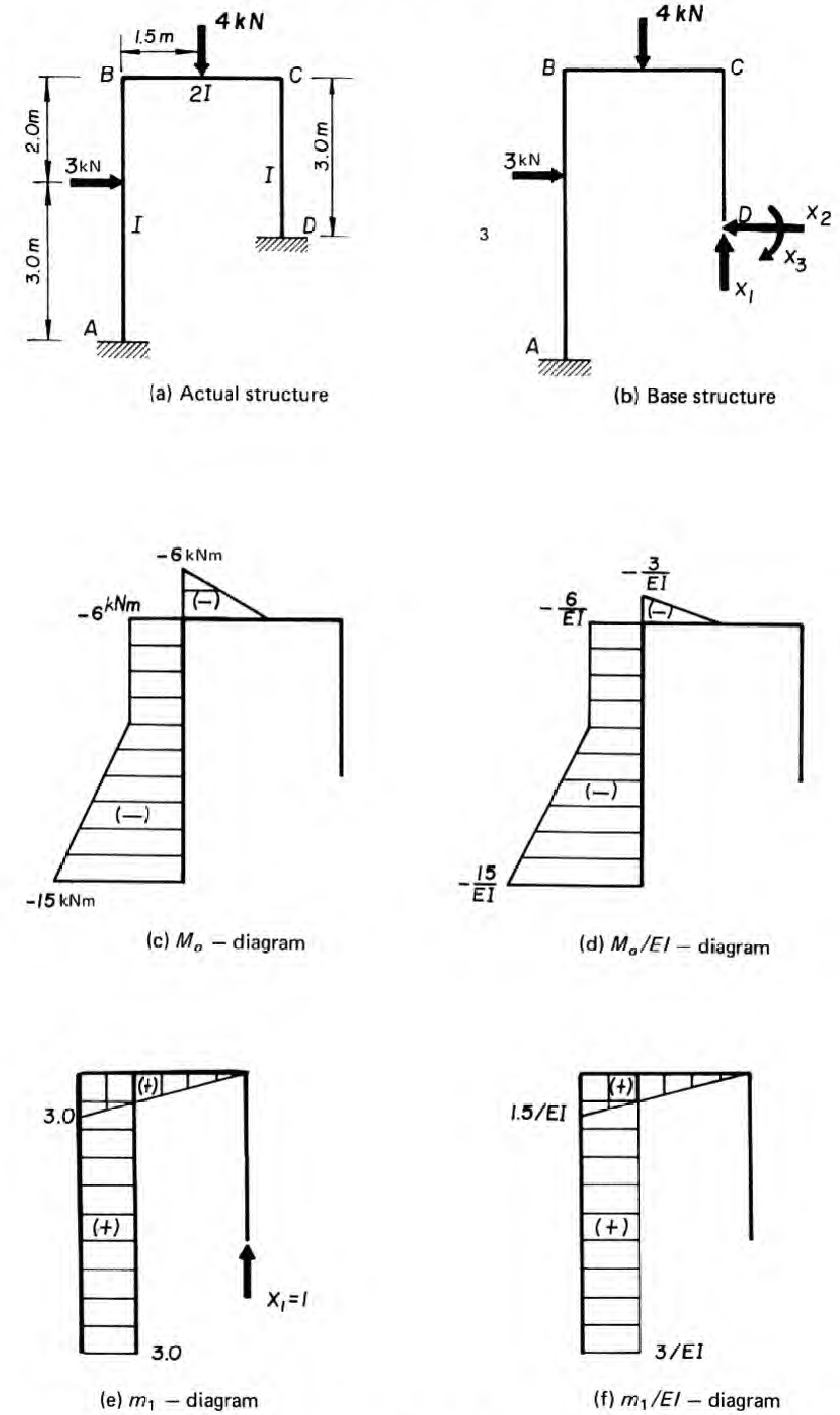
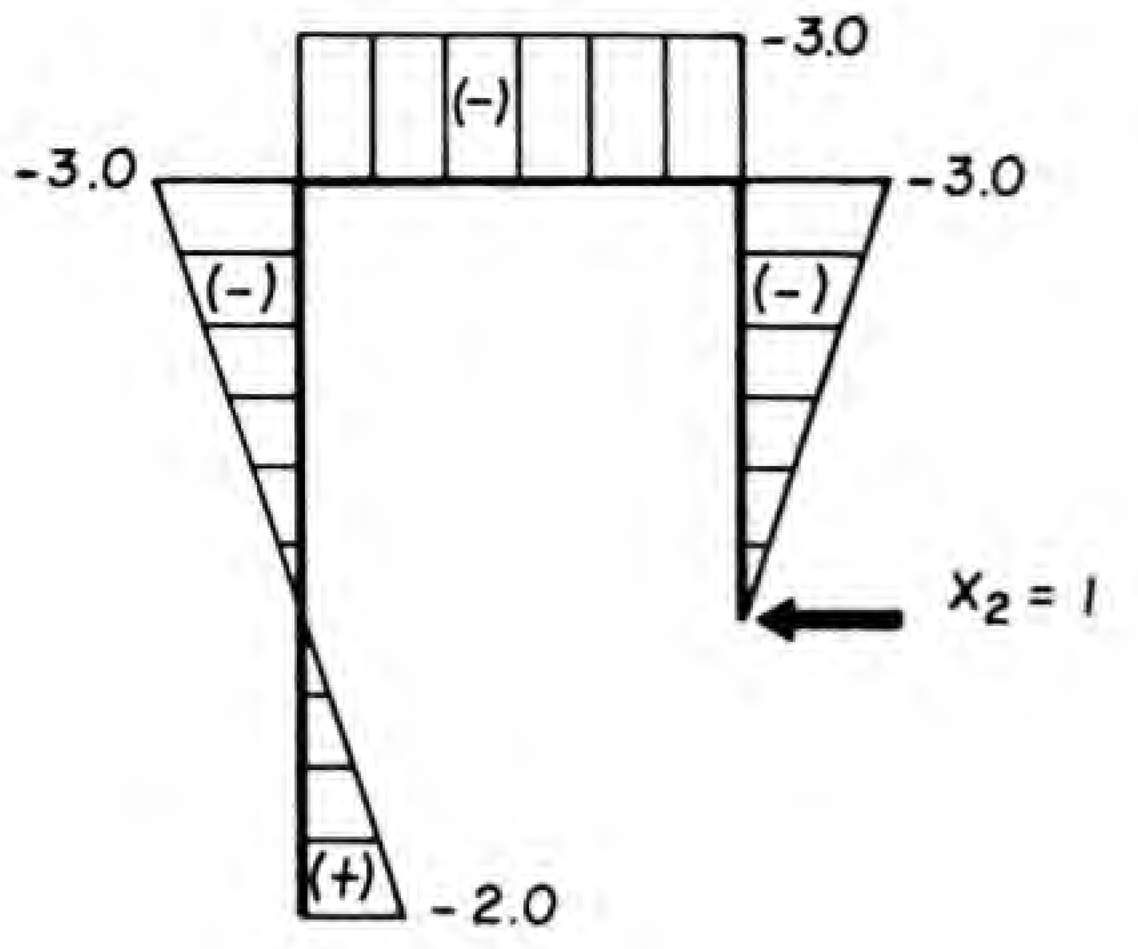
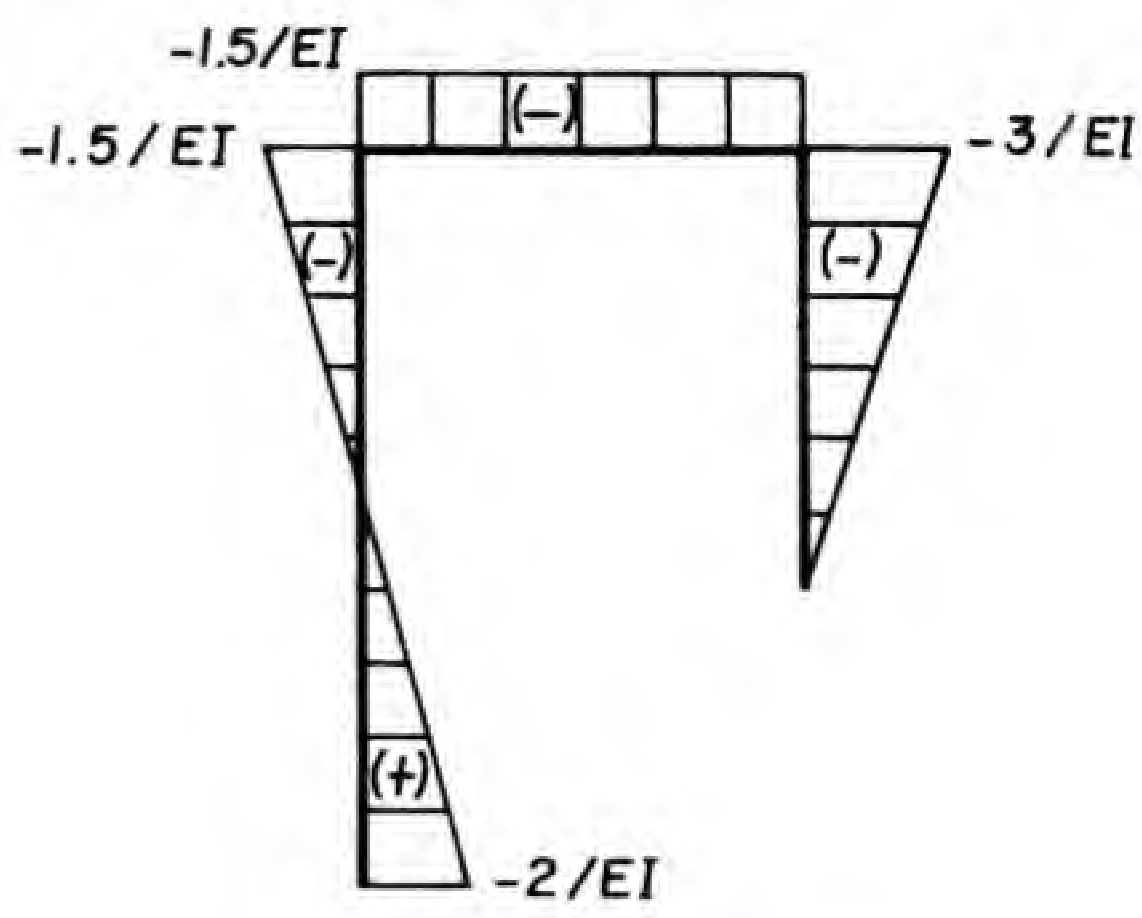

4 kN
1.5 m
B
C
2I
2.0m
3.0m
3kN
I
I
D
3.0m
A
(a) Actual structure
4 kN
B
C
3kN
D
X2
X3
X1
A
(b) Base structure
-6 kNm
-6 kNm
(–)
(–)
-15 kNm
(c) M_o – diagram
$-\frac{3}{EI}$
$-\frac{6}{EI}$
(–)
(–)
$-\frac{15}{EI}$
(d) M_o/EI – diagram
3.0
(+)
(+)
$X_1=1$
3.0
(e) m_1 – diagram
1.5/EI
(+)
(+)
3/EI
(f) m_1/EI – diagram

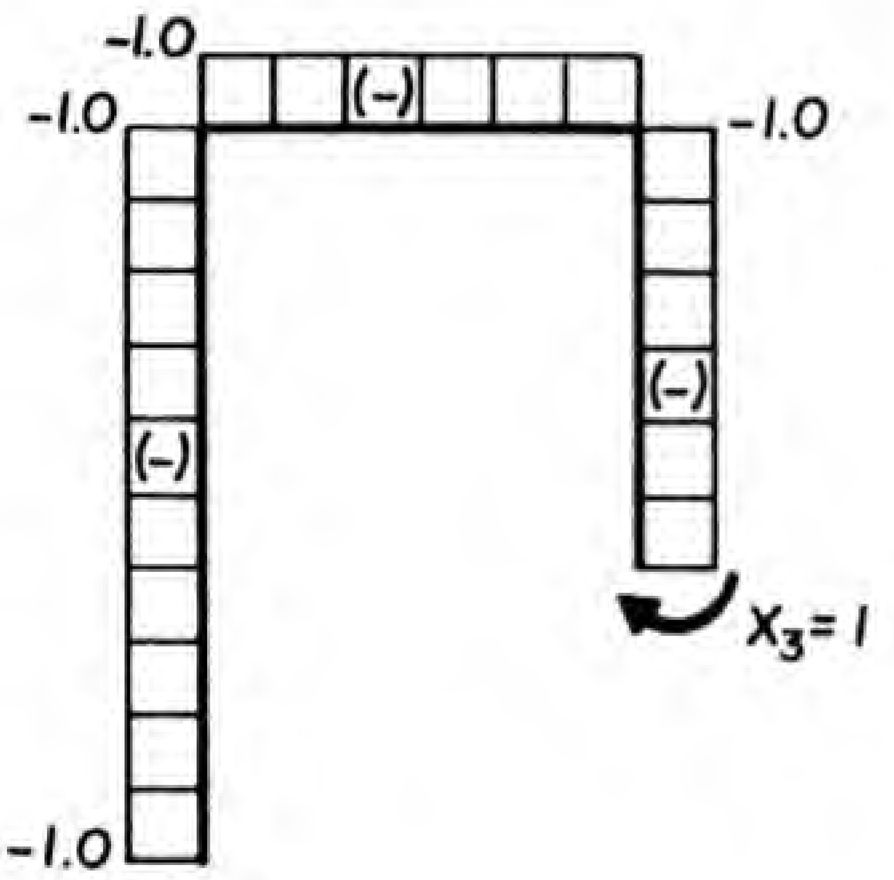
Figure 2.10



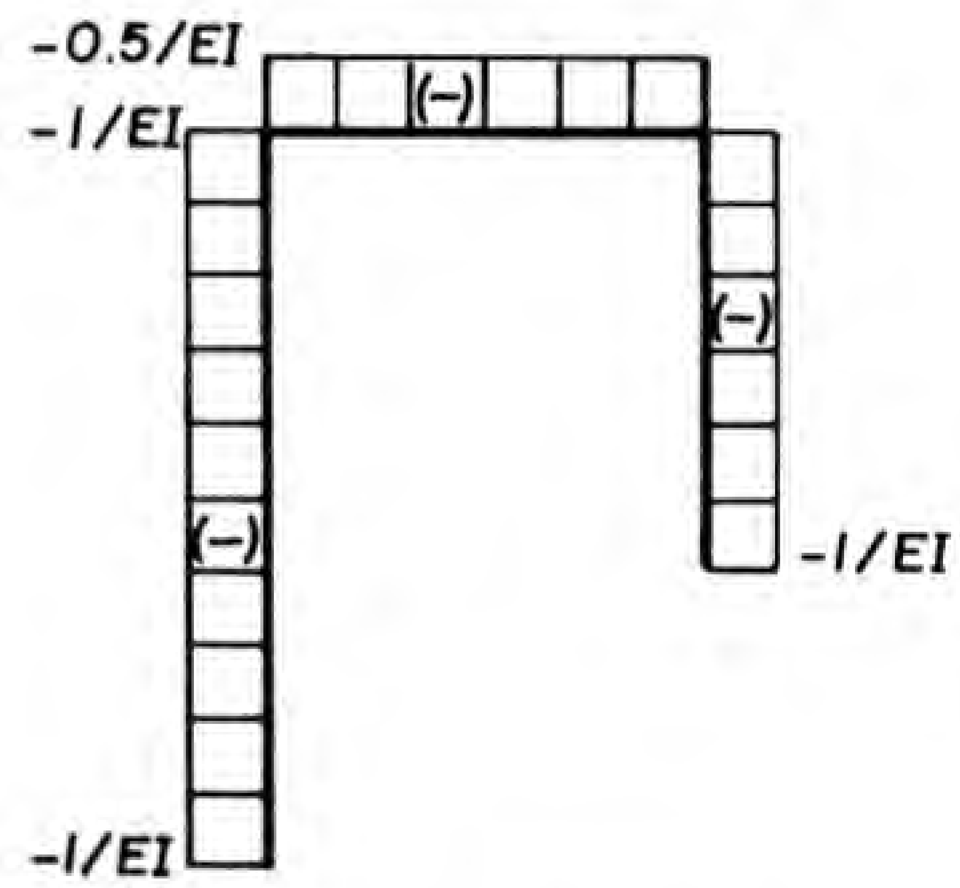
(g) m_2 – diagram



(h) m_2/EI – diagram



(i) m_3 – diagram



(j) m_3/EI – diagram

Figure 2.10 contd.

Solving the simultaneous equation:

$V_D = X_1 = 2.34$ kN (upward)

$H_D = X_2 = 1.56$ kN (to left)

$M_D = X_3 = 2.45$ kN m (counter-clockwise)

2.5 THE ELASTIC CENTRE METHOD

The elastic centre method is a special method of solving statically indeterminate structures of the one-loop form. Rigid-jointed portal frames, single-bay gabled bents, single-span arch, closed or ring structures are examples of the type of problems easily solved by this method.

Consider a fixed arch as in Fig. 2.11(a) under an arbitrary loading which produces bending moments M_0 in the primary structure.

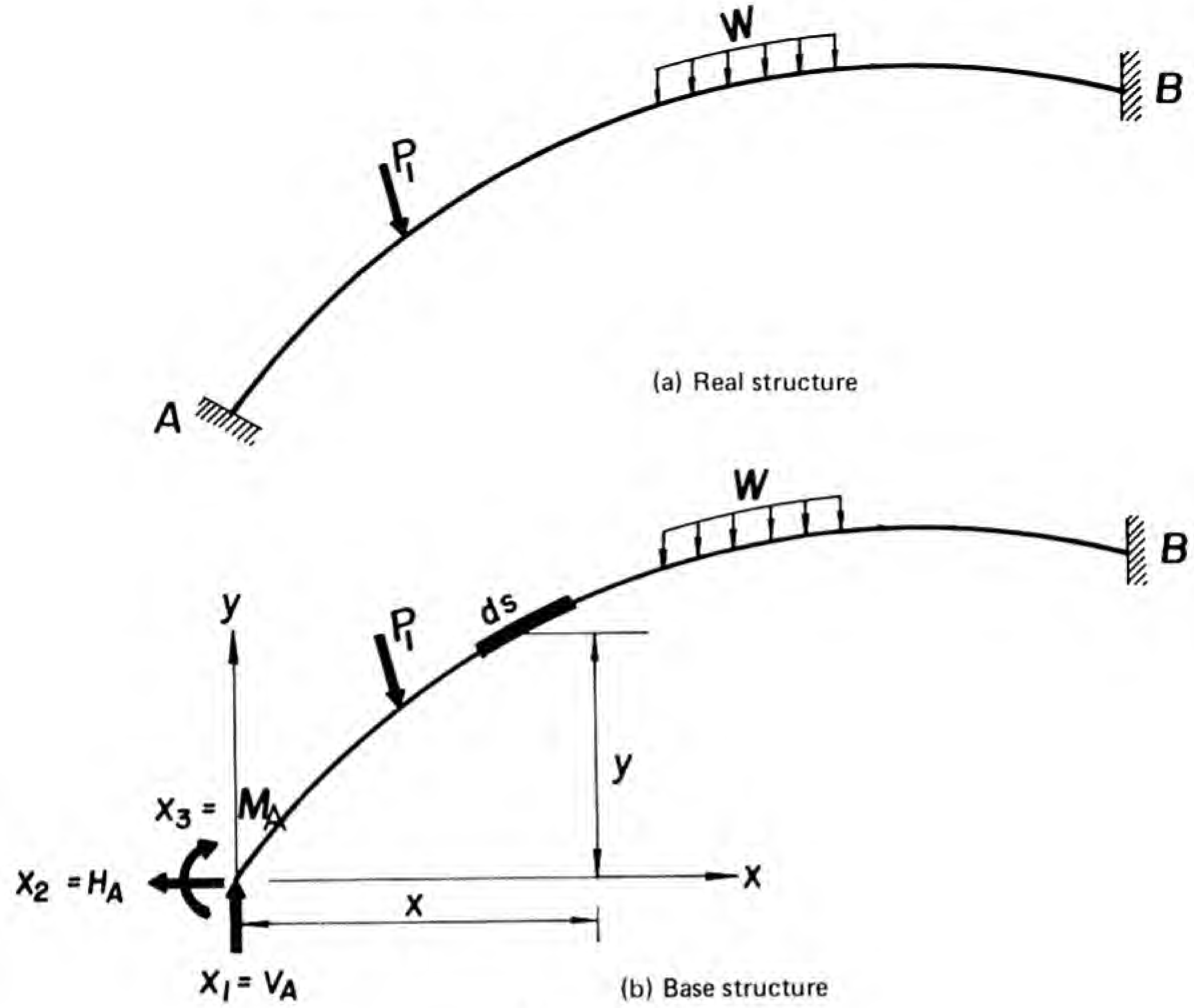


Figure 2.11

The primary structure is taken to be a cantilever as in Fig. 2.11(b), where the left support is removed and the redundant reactions $X_1 = V_A$, $X_2 = H_A$ and $X_3 = M_A$ are applied at the support point. The compatibility equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{21} & \delta_{31} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{23} & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [2.20]$$

The displacements are evaluated as

$$\Delta_{10} = \int \frac{M_o m_1 ds}{EI} = \int \frac{M_o x ds}{EI}$$

$$\Delta_{20} = \int \frac{M_o m_2 ds}{EI} = \int \frac{M_o y ds}{EI} \qquad [2.21]$$

$$\Delta_{30} = \int \frac{M_o m_3 ds}{EI} = \int \frac{M_o ds}{EI}$$

since $m_1 = x$, $m_2 = y$ and $m_3 = 1$.

Also

$$\delta_{11} = \int \frac{m_1^2 ds}{EI} = \int \frac{x^2 ds}{EI}$$

$$\delta_{22} = \int \frac{m_2^2 ds}{EI} = \int \frac{y^2 ds}{EI}$$

$$\delta_{33} = \int \frac{m_3^2 ds}{EI} = \int \frac{ds}{EI} \tag{2.22}$$

$$\delta_{12} = \delta_{21} = \int \frac{m_1 m_2 ds}{EI} = \int \frac{xyds}{EI}$$

$$\delta_{13} = \delta_{31} = \int \frac{m_1 m_3 ds}{EI} = \int \frac{xds}{EI}$$

$$\delta_{23} = \delta_{32} = \int \frac{m_2 m_3 ds}{EI} = \int \frac{yds}{EI}$$

If ds/EI is considered as an elemental area of length ds and a width normal to the arch axis of $1/EI$, then the following interpretations may be made:

δ_{11} and δ_{22} = moment of inertia of $1/EI$ area about the y axis and x axis, respectively

δ_{33} = total $1/EI$ area of the arch

δ_{12} = product of inertia of $1/EI$ about the given axis

δ_{23} = statical moment of $1/EI$ area about the y axis and x axis, respectively

If the origin of the axes can be transferred to the centroid or *elastic centre* of the *elastic area*, ds/EI, the computation may be simplified by the virtue of the fact that δ_{13} and δ_{23}, being the statical moments of elastic areas, disappear. Also if the axes through the elastic centre are the *principal axes*, δ_{12}, being the product of inertia, also vanishes.

It is statically possible to transfer the forces $X_1 = V_A$ and $X_2 = H_A$ to any point, provided $X_3 = M_A$ is properly modified, since any force may be replaced by an equal parallel force acting through any arbitrarily chosen point and a couple. Accordingly, the redundants may be applied at point 0 (x_0, y_0) which is attached to A by a perfectly rigid arm (Fig. 2.12). It is clear that this arm does not fundamentally change the structure, since, being rigid, it makes no direct contribution to the deflection of the arch.

Taking 0 as the origin of coordinates,

$$V_A = X_1$$

$$H_A = X_2 \tag{2.23}$$

$$M_A = X_3 + X_1 x_0 + X_2 y_0$$

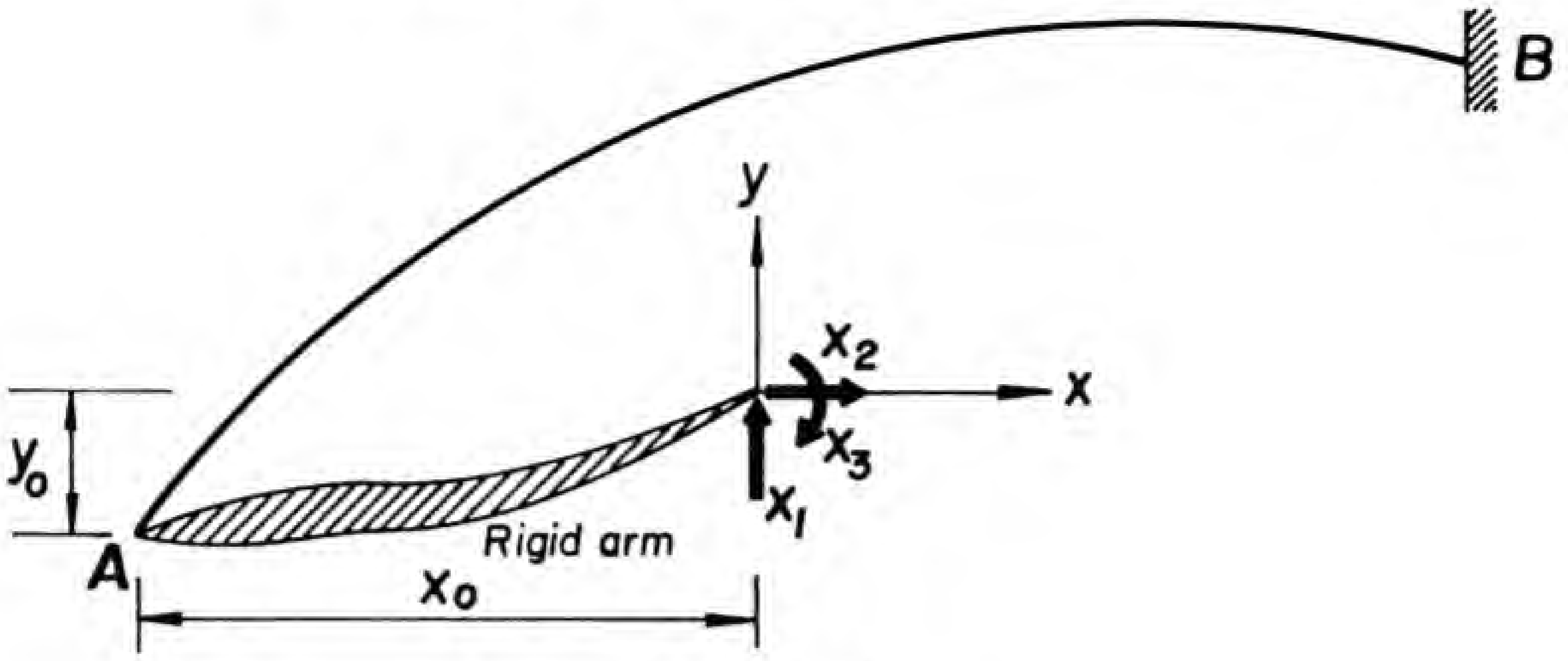


Figure 2.12

If the structure is symmetrical, the centroidal-principal axis will coincide with the axis of symmetry, and hence the product of inertia will be zero. Thus

$$\delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0$$

The corresponding compatibility equations are

$$\begin{aligned} \Delta_{10} + X_1\delta_{11} &= 0 \\ \Delta_{20} + X_2\delta_{22} &= 0 \\ \Delta_{30} + X_3\delta_{33} &= 0 \end{aligned} \qquad [2.24]$$

where the redundant reactions are taken at the elastic centre.

If the structure is not symmetrical where the x and y axes are not the principal axes, δ_{12} will not vanish, while δ_{13} and δ_{23} disappear. The compatibility equations for this case are

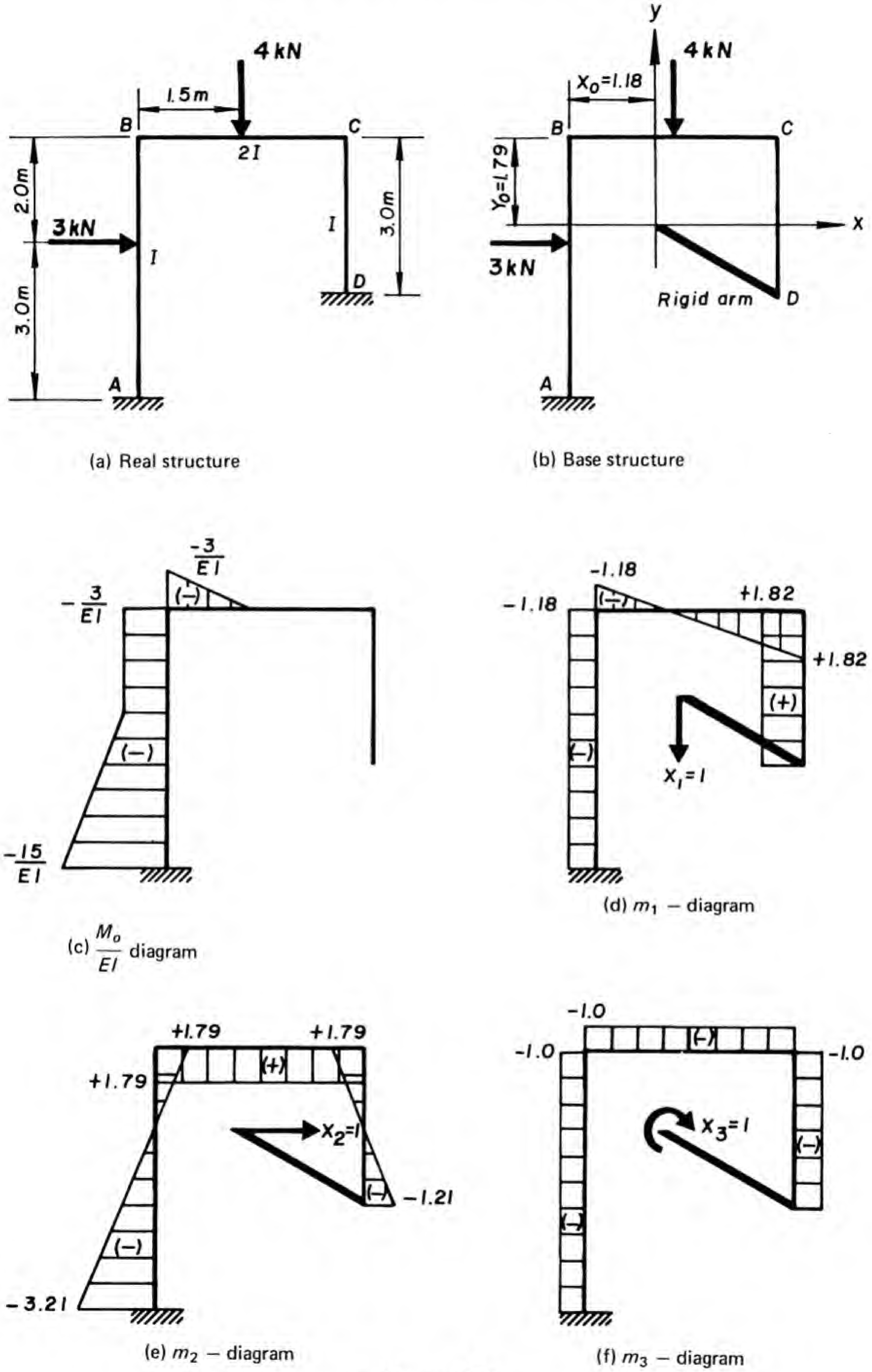
$$\begin{aligned} \Delta_{10} + X_1\delta_{11} + X_2\delta_{21} &= 0 \\ \Delta_{20} + X_1\delta_{12} + X_2\delta_{22} &= 0 \\ \Delta_{30} + X_3\delta_{33} &= 0 \end{aligned} \qquad [2.25]$$

Solving the simultaneous equations

$$\begin{aligned} X_1 &= -\frac{\Delta_{10} - (\delta_{12}/\delta_{22})\Delta_{20}}{\delta_{11} - (\delta_{12}/\delta_{22})\delta_{12}} \\ X_2 &= -\frac{\Delta_{20} - (\delta_{12}/\delta_{11})\Delta_{10}}{\delta_{22} - (\delta_{12}/\delta_{11})\delta_{12}} \\ X_3 &= -\Delta_{30}/\delta_{33} \end{aligned} \qquad [2.26]$$

EXAMPLE 2.4 *Determine the reaction components at D of the rigid frame of Fig. 2.13, using the elastic centre method.*

The elastic centre is located by taking moments about AB for the x coordinate



(a) Real structure

(b) Base structure

(c) $\frac{M_o}{EI}$ diagram

(d) m_1 – diagram

(e) m_2 – diagram

(f) m_3 – diagram

Figure 2.13

and about BC for the y coordinate:

$$x_o = \frac{\frac{(3)(3)}{1} + \frac{(3)(1.5)}{2}}{5 + \frac{3}{2} + 3} = 1.18 \text{ m}$$

$$y_o = \frac{\frac{(5)(2.5)}{1} + \frac{(3)(3)}{2}}{9.5} = 1.79 \text{ m}$$

The displacement coefficients are computed using graphic multiplication method:

$$EI\Delta_{10} = \left(\frac{-3 \times 1.5}{2}\right)(-0.68) + (-6 \times 5)(-1.18) + \left(-\frac{9 \times 3}{2}\right)(-1.18)$$

$$= 52.87$$

$$EI\Delta_{20} = \left(-\frac{3 \times 1.5}{2}\right)(1.79) + (-6 \times 5)\left(\frac{-3.21 + 1.79}{2}\right) + \left(-\frac{9 \times 3}{2}\right)(-2.2$$

$$= 47.11$$

$$EI\Delta_{30} = \left(-\frac{3 \times 1.5}{2}\right)(-1.0) + (-6 \times 5)(-1) + \left(-\frac{9 \times 3}{2}\right)(-1.0)$$

$$= 47.75$$

$$EI\delta_{11} = (-1.18 \times 5)(-1.18) + \frac{1}{2}\left(-\frac{1.18 \times 1.8}{2}\right)\left(-\frac{2}{3} \times 1.18\right)$$

$$+ \frac{1}{2}\left(-\frac{1.82 \times 1.8}{2}\right)\left(-\frac{2}{3} \times 1.18\right) + (3 \times 1.82)(1.82)$$

$$= 18.18$$

$$EI\delta_{22} = \left(-\frac{3.21 \times 3.21}{2}\right)\left(-\frac{2}{3} \times 3.21\right) + \left(\frac{1.79 \times 1.79}{2}\right)\left(\frac{2}{3} \times 1.79\right)$$

$$+ \frac{1}{2}(1.79 \times 3)(1.79) + \left(-\frac{1.79 \times 1.79}{2}\right)\left(-\frac{2}{3} \times 1.79\right)$$

$$+ \left(\frac{1.21 \times 1.2}{2}\right)\left(\frac{2}{3} \times 1.21\right)$$

$$= 20.25$$

$$EI\delta_{33} = (5 \times 1)(1) + \frac{1}{2}(3 \times 1)(1) + (3 \times 1)(1)$$

$$= 9.5$$

Since the axes through 0 are not principal axes, δ_{12} will not vanish. Thus,

$$EI\delta_{12} = -\frac{(3.21 + 1.79)}{2} \times 5\,(-1.18) + \frac{1}{2}\,\frac{(1.82 - 1.18)}{2} \times 3\,(1.79)$$

$$+\frac{1.79 - 1.21}{2} \times 3\,(1.82)$$

$$= 6.63$$

The elastic equations are

$$\begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & 0 \\ \delta_{12} & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

substituting

$$\begin{bmatrix} 52.87 \\ 47.11 \\ 47.75 \end{bmatrix} + \begin{bmatrix} 18.18 & 6.63 & 0 \\ 6.63 & 20.25 & 0 \\ 0 & 0 & 9.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of the simultaneous equations is

$$X_1 = -2.34 \text{ kN}$$

$$X_2 = -1.56 \text{ kN}$$

$$X_3 = -4.82 \text{ kN m}$$

The values of the redundants at support D are:

$$V_D = 2.34 \text{ kN (upward)}$$

$$H_D = 1.56 \text{ kN (left)}$$

$$M_D = 4.82 - 2.34 \times 1.82 + 1.56 \times 1.21$$

$$= 2.45 \text{ kN m (counter-clockwise)}$$

2.6 THE THREE-MOMENT EQUATIONS

Consider a continuous beam with n spans as shown in Fig. 2.14. This beam is indeterminate to the $(n - 1)$th degree when the support reactions are taken as the redundants, each of which contain all the unknowns. However, when support moments are used as the redundants, although the same number of equations must eventually be solved, each equation contains only three of the unknowns. The latter choice of redundants localises the loading conditions on

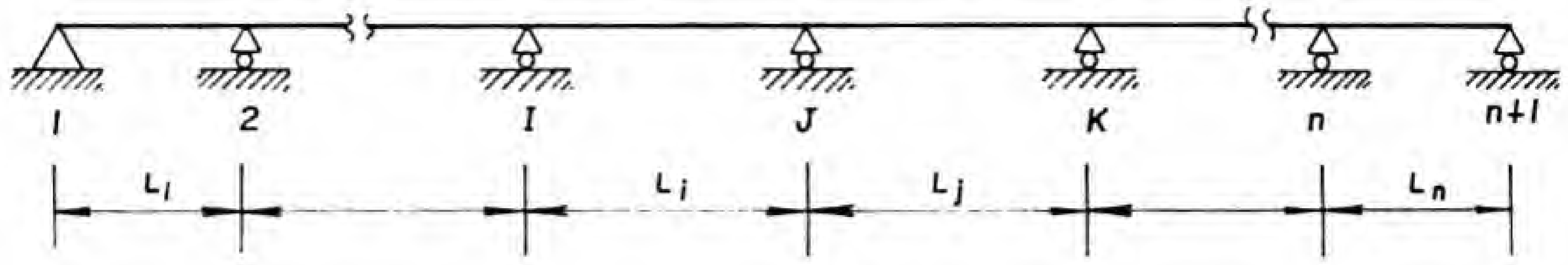


Figure 2.14

the base structure and the resulting relationship between the redundants permits the equations to be written in a simple and systematic manner. These equations express the three-moment equations first presented by the French engineer Clapeyron.

The three-moment equation expresses the relation between the bending moments at the three successive supports of a continuous beam. The relation is derived from the continuity of the elastic curve when the compatibility equations are obtained in terms of the support slopes of adjacent spans.

Consider two adjacent spans IJ and JK of a continuous beam shown in Fig. 2.15. The moment of inertia is considered constant between I and J and equal to I_i and likewise constant between J and K and equal to I_j. The beam is assumed to be initially straight, and the support settlements amounting to Δ_i, Δ_j and Δ_k take place at support I, J and K respectively, as indicated by the heavy line in Fig. 2.15(e).

Compatibility equations are written at each support expressing the equality of end slopes at adjacent spans. The condition of continuity of the slope gives

$$\frac{\Delta_j - \Delta_i + \delta_i}{L_i} = \frac{\Delta_k - \Delta_j - \delta_k}{L_j}$$

Rearrange the equation as

$$\frac{\delta_i}{L_i} + \frac{\delta_k}{L_j} = \frac{\Delta_i - \Delta_j}{L_i} + \frac{\Delta_k - \Delta_j}{L_i} \qquad [2.27]$$

But from the Second Theorem of the Area-moment Method,

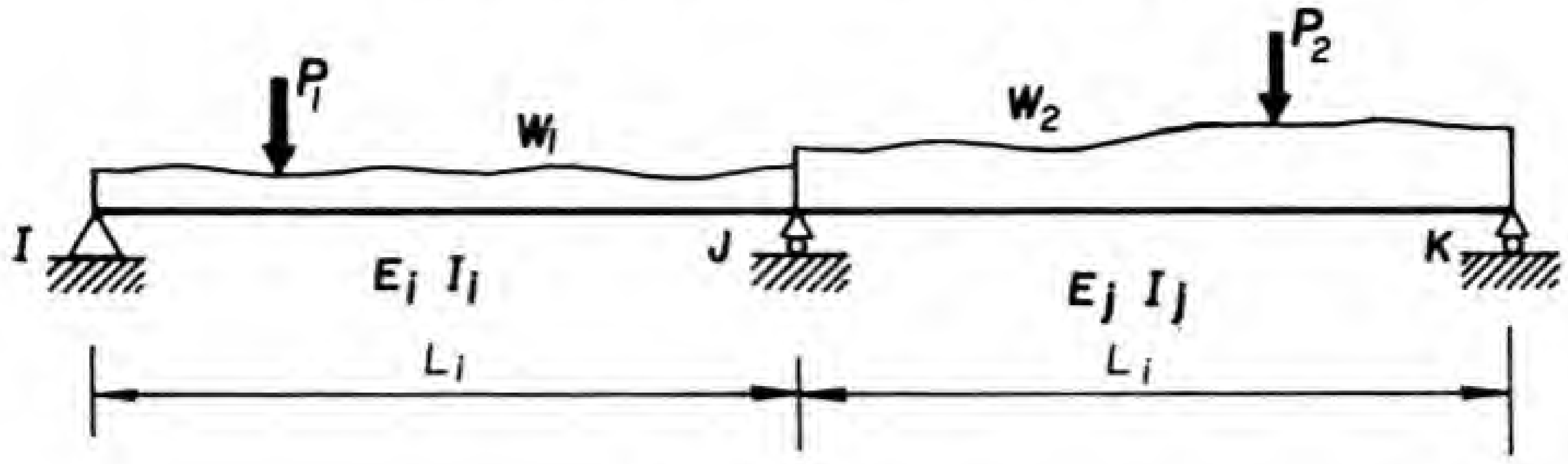
$$\delta_i = \frac{1}{E_i I_i}\left[A_i\bar{x}_i + \left(\frac{1}{2}M_i L_i\right)(L_i/3) + \left(\frac{1}{2}M_j L_i\right)(2L_i/3)\right]$$

$$\delta_k = \frac{1}{E_j I_j}\left[A_j\bar{x}_j + \left(\frac{1}{2}M_k L_j\right)(L_j/3) + \left(\frac{1}{2}M_j L_j\right)(2L_j/3)\right] \qquad [2.28]$$

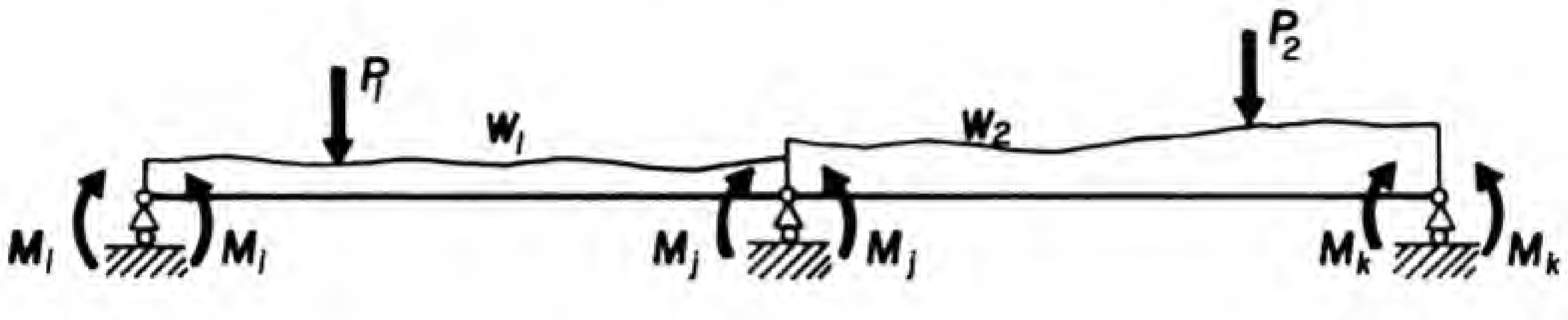
Combining [2.27] and [2.28] gives Clapeyron's Equation of Three Moments:

$$M_i\left(\frac{L_i}{E_i I_i}\right) + 2M_j\left(\frac{L_i}{E_i I_i} + \frac{L_j}{E_j I_j}\right) + M_k\left(\frac{L_j}{E_j I_j}\right)$$

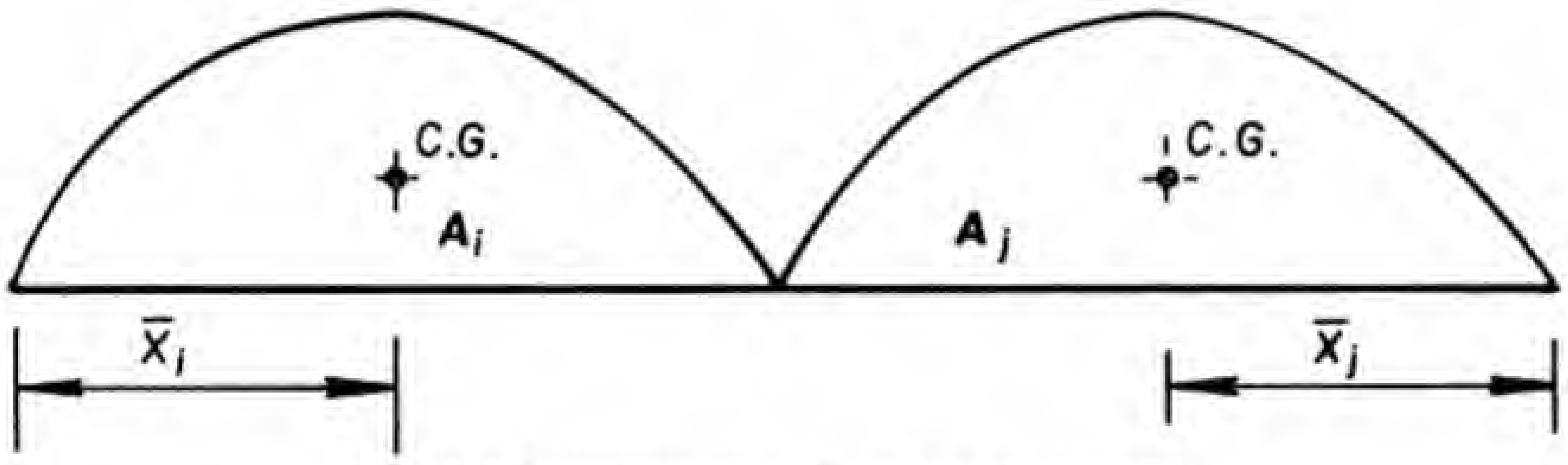
$$+ 6\left(\frac{A_i\bar{x}_i}{E_i I_i L_i} + \frac{A_j\bar{x}_i}{E_j I_j L_j}\right) = 6\left(\frac{\Delta_i - \Delta_j}{L_i} + \frac{\Delta_k - \Delta_j}{L_j}\right) \qquad [2.29]$$



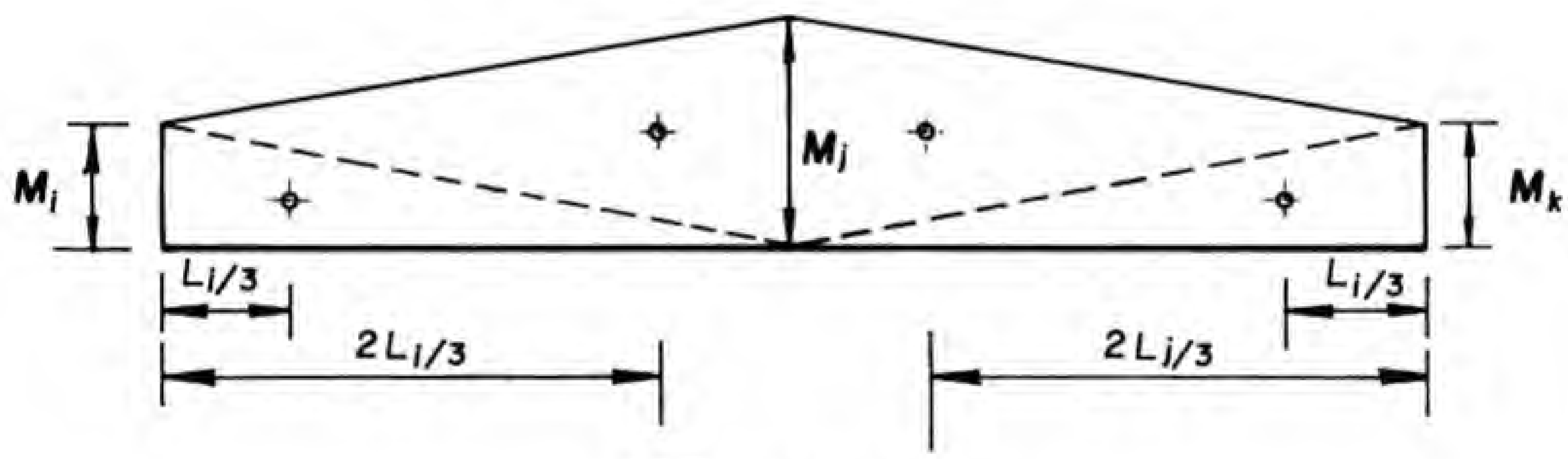
(a) Actual beam



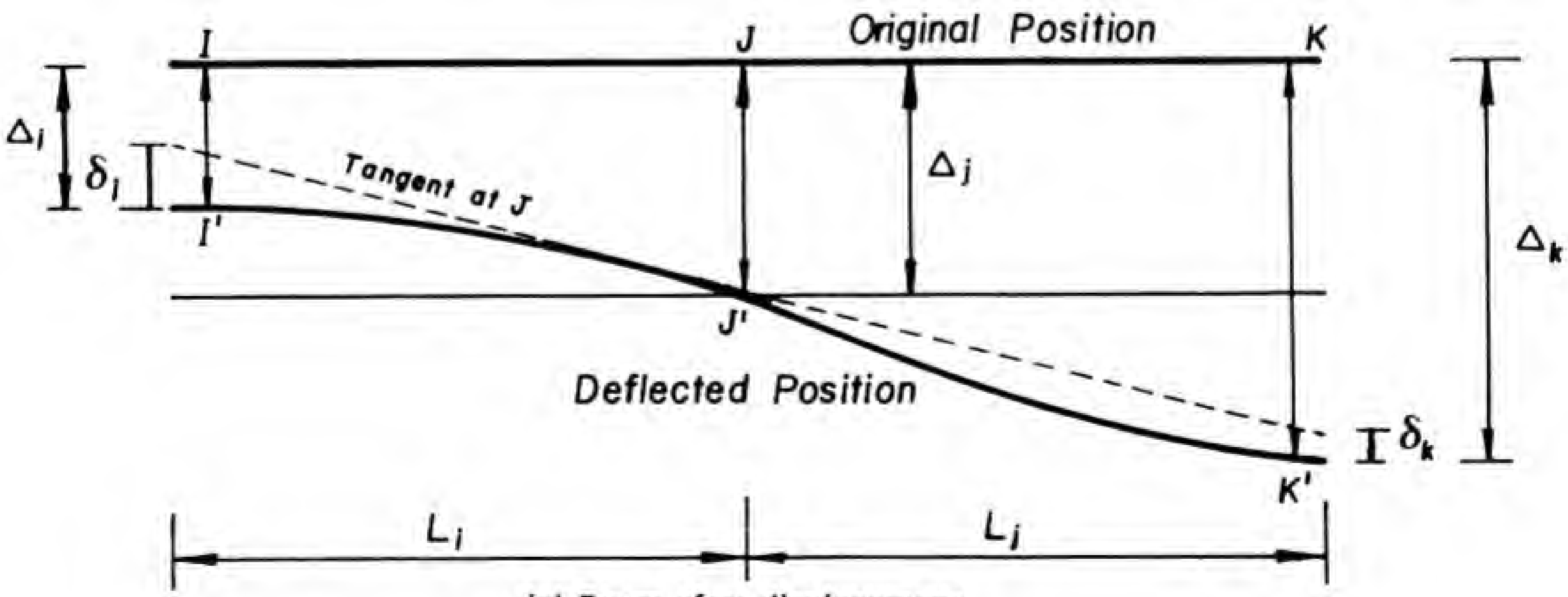
(b) Base structure



(c) Base structure bending moment



(d) Support moments



(e) Beam after displacement

Figure 2.15

In the particular case when $I_i = I_j = I$ and there are no support settlements, the equation becomes

$$M_i L_i + 2M_j(L_i + L_j) + M_k L_j = -6\frac{A\bar{x}}{L} \qquad [2.30]$$

In the special case when the continuous beam has a fixed support, the bending moment at the fixed support is a redundant for which a new equation must be established. This case may be treated by considering the fixed support equivalent to an outer imaginary beam of finite span with infinite stiffness. This is illustrated in Fig. 2.16.

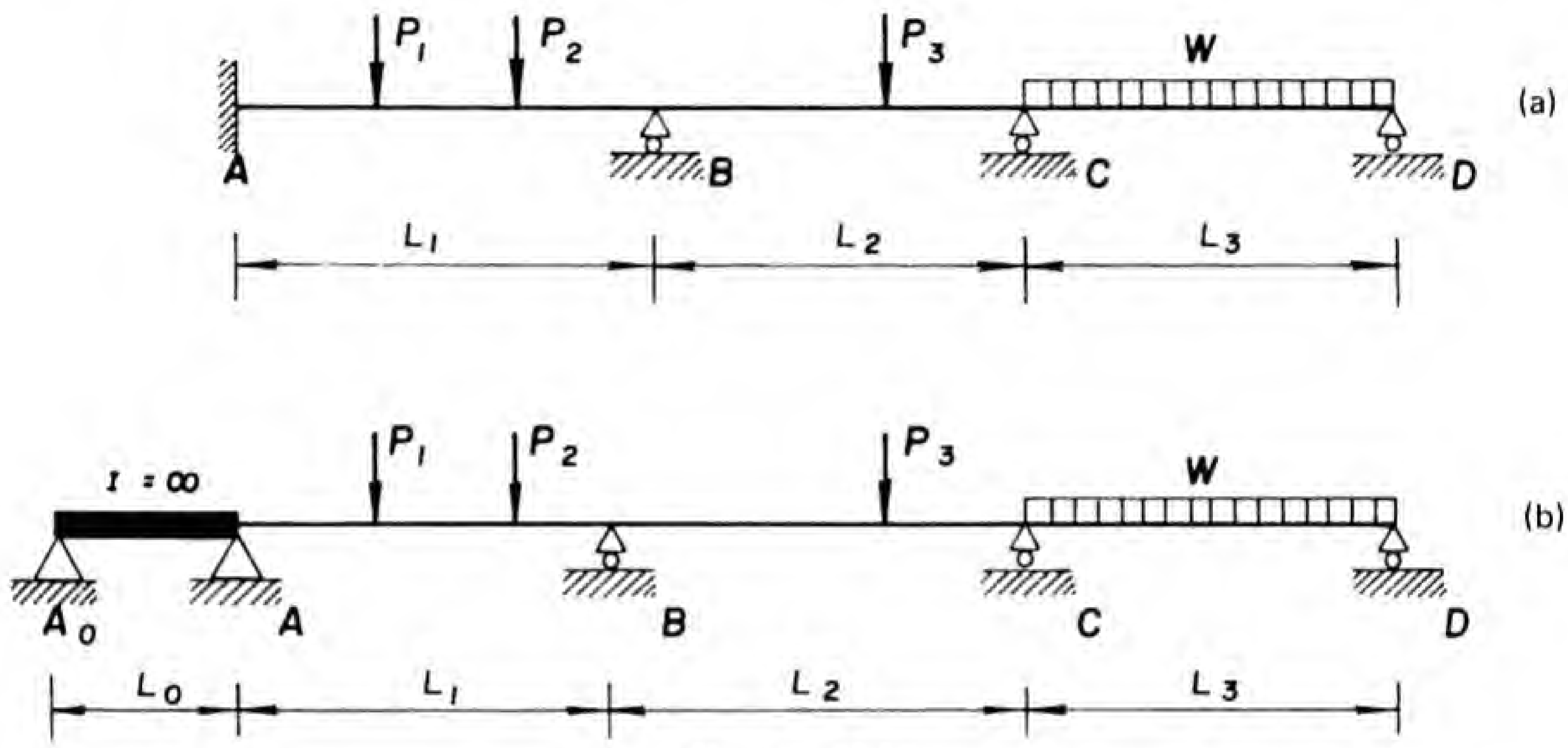


Figure 2.16

The three-moment equation, when applied to span A_0A and AB is

$$0 + 2M_A\left(\frac{L_0}{\infty} + \frac{L}{I}\right) + M_B\left(\frac{L}{I}\right) = -\Sigma\frac{6A\bar{x}}{LI}$$

This reduces to

$$2M_A L + M_B L = -\Sigma\frac{6A\bar{x}}{L}$$

EXAMPLE 2.5 *Find the support moments of the continuous beam shown in Fig. 2.17 using the three-moment equations.*

The three-moment equations are written for:

Span A_0A and AB

$$M_{A_0}\left(\frac{L_0}{\infty}\right) + 2M_A\left(\frac{L_0}{\infty} + \frac{4}{2}\right) + M_B\left(\frac{4}{2}\right) = -\frac{6 \times 12 \times 7/3}{2 \times 4}$$

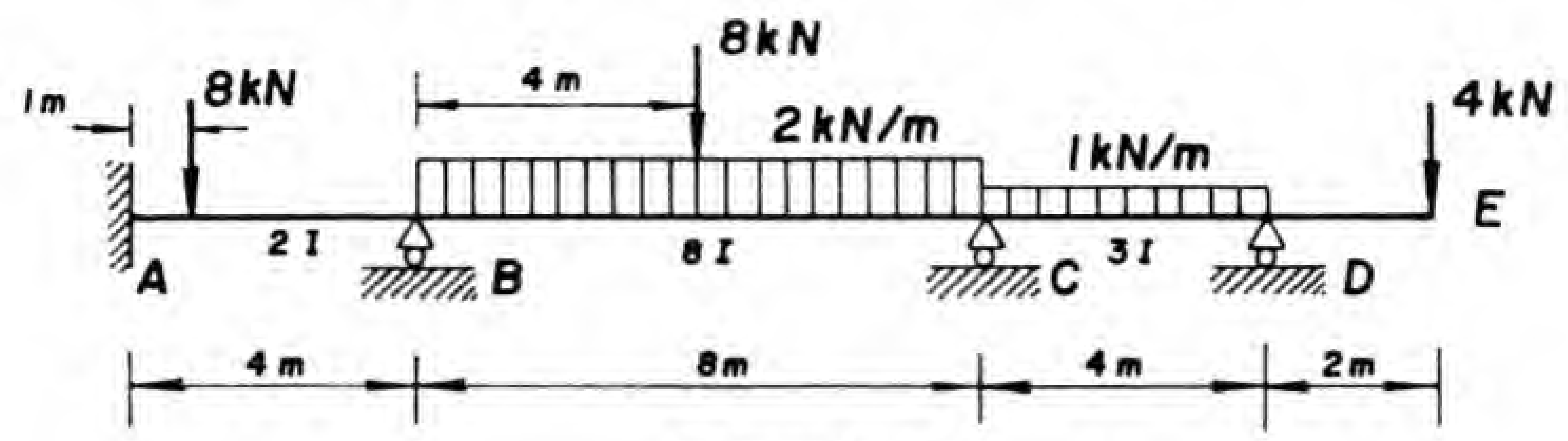


Figure 2.17

Span AB and BC

$$M_A\left(\frac{4}{2}\right) + 2M_B\left(\frac{4}{2}+\frac{8}{8}\right) + M_C\left(\frac{8}{8}\right) = -\frac{[(6)(12)\times 5]3}{(2)(4)} - \frac{(6)(149.33)4}{(8)(8)}$$

Span BC and CD

$$M_B\left(\frac{8}{8}\right) + 2M_C\left(\frac{8}{6}+\frac{6}{3}\right) + M_D\left(\frac{6}{3}\right) - \frac{(6)(149.33)4}{(8)(8)} - \frac{(6)(18)3}{(3)(6)}$$

Simplifying:

$$2M_A + M_B = -10.5$$

$$2M_A + 6M_B + M_C = -71$$

$$M_B + 6M_C = -58$$

Solving:

$$M_A = +0.01 \text{ kN m}$$

$$M_B = -10.52 \text{ kN m}$$

$$M_C = -7.91 \text{ kN m}$$

2.7 THE METHOD OF ELASTIC WORK

Consider the indeterminate beam shown in Fig. 2.18 which has two redundant reactions.

Taking the base structure as a simply supported beam, as shown in Fig. 2.18(b), subjected to the loads and the redundant reactions M_A and V_B, it is obvious that the base structure is statically equivalent to the actual structure. Compatibility condition furnishes two additional equations, namely, the rotation at A and zero. These displacements by Castigliano's First Theorem are

$$\theta_A = \frac{\partial U}{\partial M_A} = 0$$

$$\Delta_B = \frac{\partial U}{\partial V_B} = 0 \qquad [2.31]$$

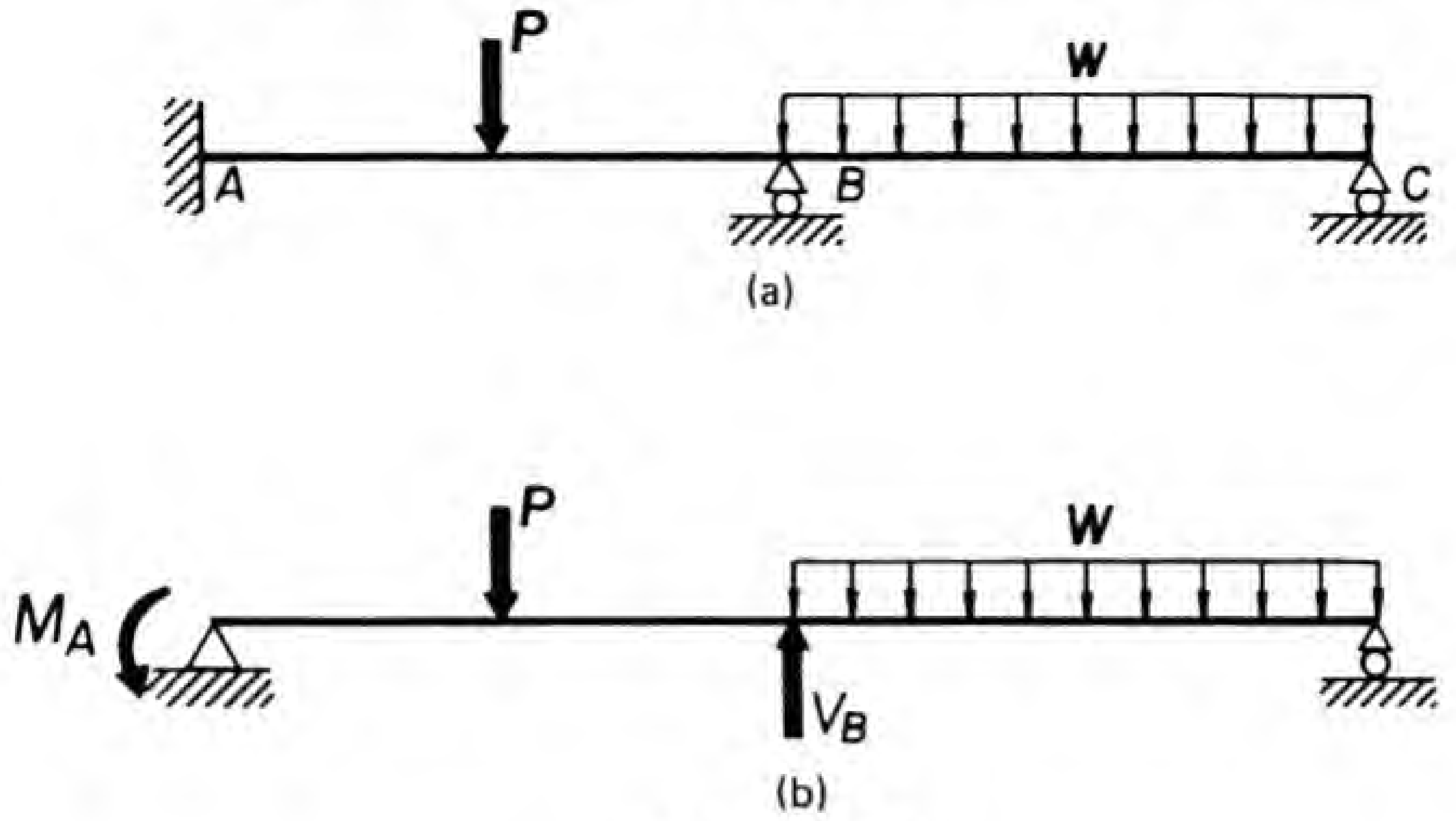


Figure 2.18

Therefore, the condition for determining the redundants M_A and V_B satisfying the displacement conditions is such that the total internal energy will be a minimum, or strictly speaking a maximum. It is evident that it must be a minimum value, because there is no maximum value when the stored energy increases, as the redundant forces increase indefinitely. The above equations may also be interpreted as follows: Among all possible sets of values that redundant forces in the system may assume, the correct set of values is that which makes the strain energy U a minimum. Since one equation is obtained for each redundant force, a set of equations corresponding to the compatibility will result.

Therefore, Castigliano's Theorem of Least Work, also commonly known as Castigliano's Second Theorem, may be stated as follows:

> In any loaded statically indeterminate structure with no temperature changes or support displacement, among all possible equilibrium states the correct values of the redundants are those for which the total elastic strain energy, resulting from the application of a given system of loads, has a minimum value.

Consider the beam shown in Fig. 2.18. The bending moment at any point on the beam is given by

$$M = M_o + M_A m_A + V_B m_B \qquad [2.32]$$

where M_o = moment in the base structure due to the applied loads

m_A = moment resulting from a unit couple at A

m_B = moment resulting from a unit vertical load at B

Using Casligliano's Theorem of Least Work:

$$\theta_{\mathrm{A}} = \frac{\partial U}{\partial M_{\mathrm{A}}} = M \int \frac{\partial M}{\partial M_{\mathrm{A}}} \frac{dx}{EI}$$

$$= \int M_o m_{\mathrm{A}} \frac{dx}{EI} + M_{\mathrm{A}} \int m_{\mathrm{A}}^2 \frac{dx}{EI} + R_{\mathrm{B}} \int m_{\mathrm{A}} m_{\mathrm{B}} \frac{dx}{EI} = 0$$

$$\Delta_{\mathrm{B}} = \frac{\partial U}{\partial V_{\mathrm{B}}} = \int M \frac{\partial M}{\partial V_{\mathrm{B}}} = \int m \frac{\partial M}{\partial V_{\mathrm{B}}} \frac{dx}{EI} \tag{2.33}$$

$$= \int M_o m_{\mathrm{B}} \frac{dx}{EI} + M_{\mathrm{A}} \int m_{\mathrm{B}} m_{\mathrm{A}} \frac{dx}{EI} + V_{\mathrm{B}} \int m_{\mathrm{B}}^2 \frac{dx}{EI} = 0$$

From the above two equations, the integrals are evaluated for the base structure and the redundants M_{A} and R_{B} are then solved.

Writing the above equations in terms of displacements:

$$\theta_{\mathrm{A}} = \Delta_a + M_{\mathrm{A}}\delta_{aa} + V_{\mathrm{B}}\delta_{ab} = 0$$

$$\Delta_{\mathrm{B}} = \Delta_b + M_{\mathrm{A}}\delta_{ba} + V_{\mathrm{B}}\delta_{bb} = 0 \tag{2.34}$$

Note that these equations are identical in form with those obtained from the virtual work method.

EXAMPLE 2.6 *Find the reactions of the propped cantilever beam shown in Fig. 2.19 using Castigliano's Theorem of Least Work.*

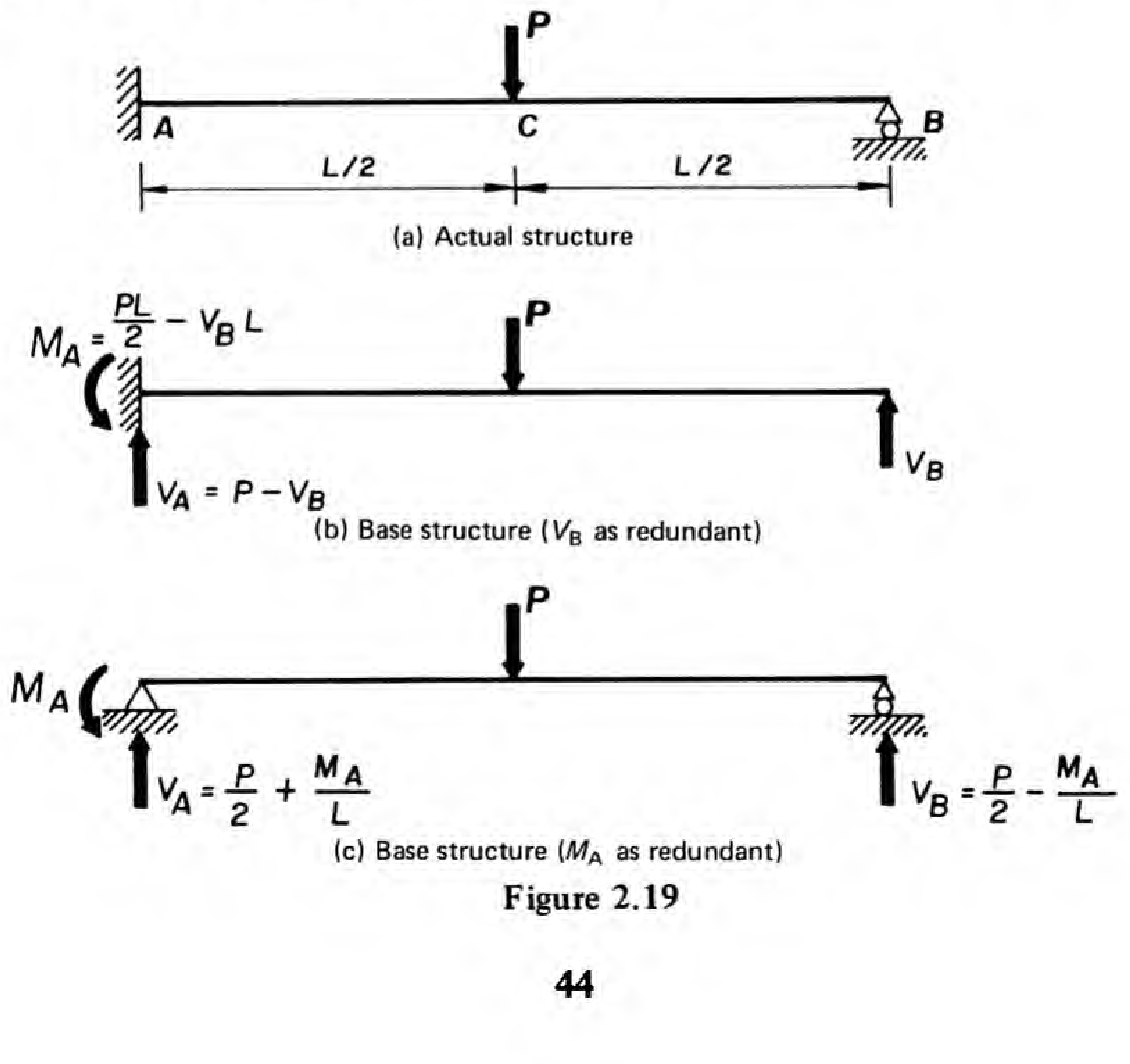


Figure 2.19

(a) Taking the reaction at B, V_B, as the redundant (Fig. 2.19(b)), the redundant V_B is determined from

$$\frac{\partial U}{\partial V_B} = 0$$

Table 2.2

Section	Origin $x = 0$ at	Limits	M	$\partial M/\partial V_B$
AC	C	$x = 0$ to $x = L/2$	$V_B\left(x + \frac{L}{2}\right) - Px$	$\left(x + \frac{L}{2}\right)$
CB	B	$x = 0$ to $x = L/2$	$V_B x$	x

$$\frac{\partial U}{\partial V_B} = \int M \frac{\partial M}{\partial V_B}\frac{dx}{EI}$$

$$= \frac{1}{EI}\int_0^L V_B\left(x + \frac{L}{2}\right) - Px \left(x + \frac{L}{2}\right) dx$$

$$+ \frac{1}{EI}\int_0^L (V_B x)x \, dx$$

$$= \frac{1}{EI}\left[\frac{1}{3} V_B\left(x + \frac{L}{2}\right)^3 - \frac{Px^3}{3} - \frac{PLx^2}{4}\right]_0^{L/2}$$

$$+ \frac{1}{EI}\left[\frac{V_B x^3}{3}\right]_0^{L/2}$$

$$= \frac{1}{EI}\left[\frac{V_B L^3}{3} - \frac{5PL^3}{48}\right] = 0$$

Thus

$$V_B = \frac{5}{16}P$$

The other reactions are

$$R_A = P - V_B = \frac{11}{16}P$$

$$M_A = \frac{PL}{2} - V_B L = \frac{3}{16}PL \text{ (negative moment)}$$

(b) Taking M_A as the redundant (Fig. 2.19(c)):

Table 2.3

Section	Origin $x = 0$ at	Limits	M	$\partial M/\partial M_A$
AC	A	$x = 0$ to $x = L/2$	$-M_A + \left(\frac{P}{2} + \frac{M_A}{L}\right)x$	$-1 + \frac{x}{L}$
CB	B	$x = 0$ to $x = L/2$	$\left(\frac{P}{2} - \frac{M_A}{L}\right)x$	$-\frac{x}{L}$

$$\frac{\partial U}{\partial M_A} = \frac{1}{EI}\int_0^L \left[\left(-M_A + \left(\frac{P}{2} + \frac{M_A}{L}\right)x\right)\left(-1 + \frac{x}{L}\right)\right] dx$$

$$+ \frac{1}{EI}\int_0^L \left[\left(\frac{P}{2} - \frac{M_A}{L}\right)x \left(-\frac{x}{L}\right)\right] dx = 0$$

$$= \frac{1}{EI}\int_0^L \left[\left(M_A - \frac{M_A x}{L} - \frac{P}{2}x - \frac{M_A}{L}x + \left(\frac{P}{2} + \frac{M_A}{L}\right)\frac{x^2}{L}\right)\right] dx$$

$$+ \frac{1}{EI}\int_0^L \left(-\frac{P}{2} - \frac{M_A}{L}\right)\frac{x^2}{L}\, dx$$

$$= \frac{1}{EI}\left[\left(\frac{M_A L}{3} - \frac{PL^2}{16}\right)\right] = 0$$

$$M_A = \frac{3PL}{16} \quad \text{(negative moment)}$$

The other reactions are

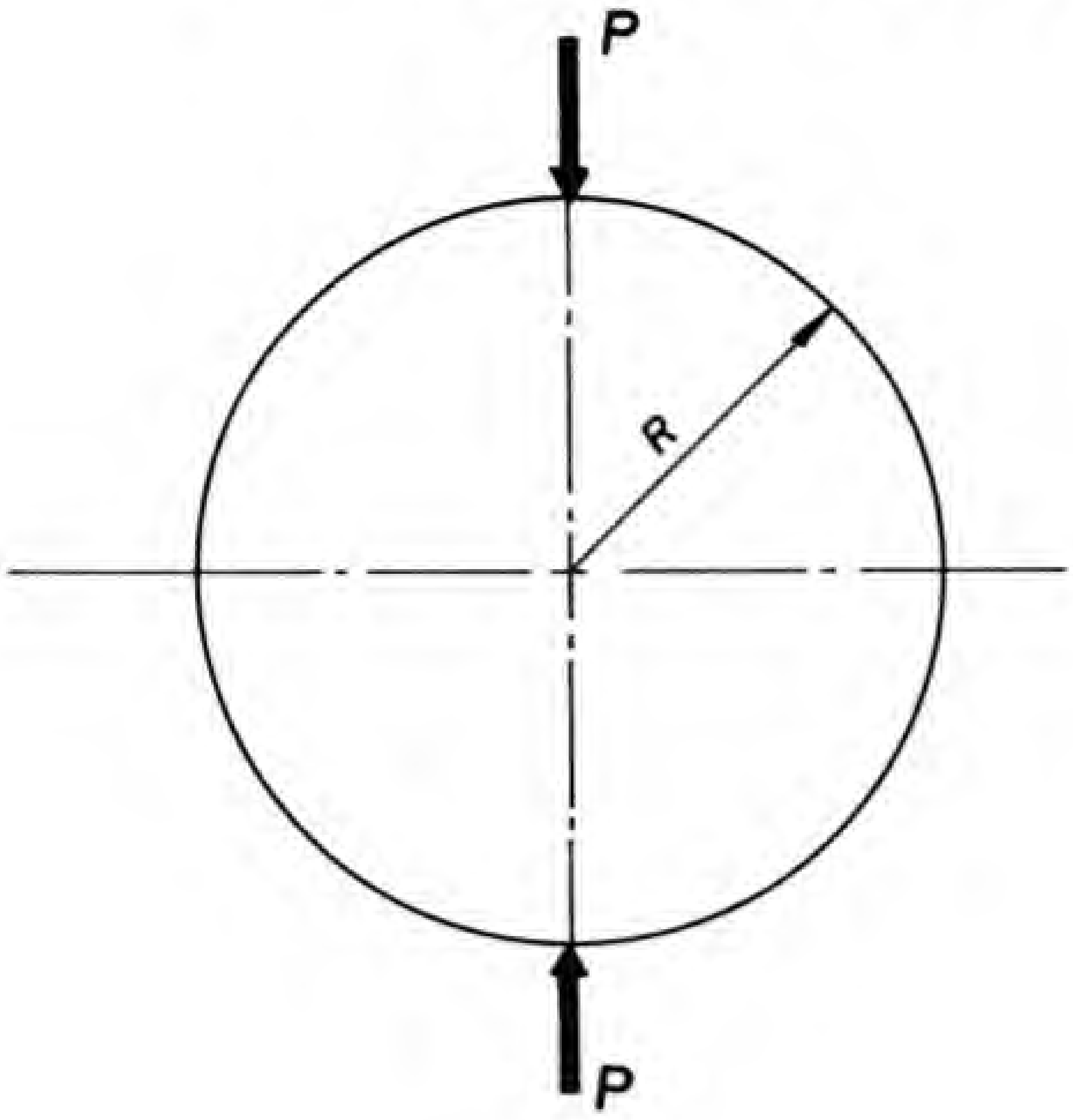
$$V_A = \frac{P}{2} + \frac{M_A}{L} = \frac{11}{16}P$$

$$V_B = \frac{P}{2} - \frac{M_A}{L} = \frac{5}{16}P$$

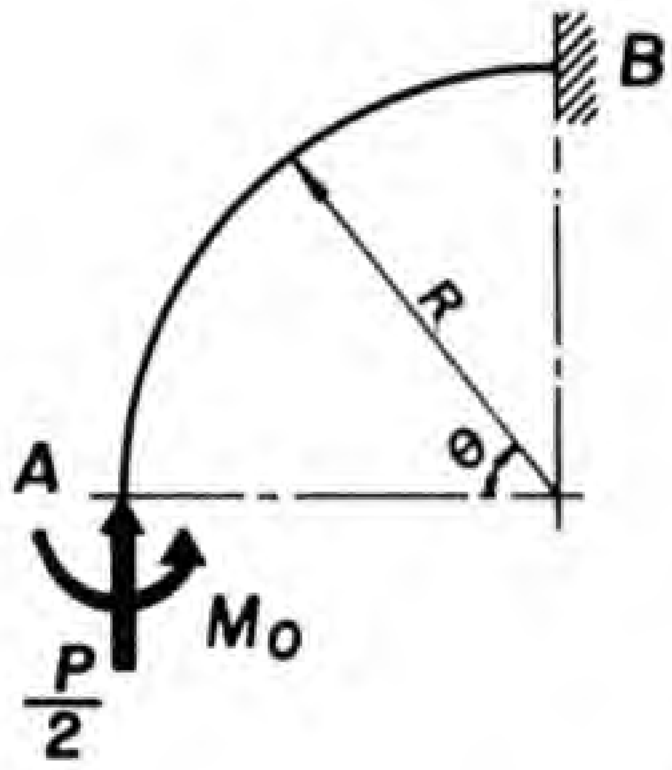
Note that a moment that causes tension on the top side of the beam is considered negative moment. The positive value obtained for M_A indicates that the assumed negative moment is correct.

EXAMPLE 2.7 *Find the maximum moment and the decrease in the vertical diameter of the ring shown in Fig. 2.20.*

Due to symmetry, one-quarter of the ring is considered.



(a) Actual structure



(b) Equivalent structure

Figure 2.20

(a) Maximum Moment

$$\theta_A = \frac{\partial U}{\partial M_o} = 0$$

$$\frac{\partial U}{\partial M_o} = \frac{M}{EI} \int \frac{\partial M}{\partial M_o} dx$$

$$M = \frac{PR}{2}(1 - \cos\theta) - M_o$$

$$\frac{\partial M}{\partial M_o} = -1$$

$$dx = R d\theta$$

Substituting,

$$\frac{\partial U}{\partial M_o} = -\frac{1}{EI}\int_0^{\pi/2}\left(\frac{PR}{2}(1-\cos\theta) - M_o\right) Rd\theta = 0$$

$$= \frac{1}{EI}\left[-\frac{PR^2}{2}\theta + \frac{PR^2}{2}\sin\theta + RM_o\theta\right]_0^{\pi/2}$$

$$M_o = \frac{PR}{2}\left(1 - \frac{2}{\pi}\right)$$

The moment at any point is

$$M = \frac{PR}{2}(1-\cos\theta) - M_o$$

$$= \frac{PR}{2}\left(\frac{2}{\pi} - \cos\theta\right)$$

The maximum moment is at the point of load application, i.e. when $\theta = \pi/2$

$$M_{\max} = \frac{PR}{\pi}$$

(b) Deflection

$$\Delta_B = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P}\right) dx$$

$$M = \frac{PR}{2}\left(\frac{2}{\pi} - \cos\theta\right)$$

$$\frac{\partial M}{\partial P} = \frac{R}{2}\left(\frac{2}{\pi} - \cos\theta\right)$$

$$dx = Rd\theta$$

Substituting:

$$\Delta_B = \frac{4}{EI}\int_0^{\pi/2}\frac{PR^2}{4}\left(\frac{2}{\pi} - \cos\theta\right)^2 Rd\theta$$

$$= \frac{PR^3}{EI}\left[\frac{4}{\pi^2}\theta - \frac{4}{\pi}\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$\Delta = \frac{PR^3}{EI}\left(\frac{\pi}{4} - \frac{2}{\pi}\right)$$

Castigliano's Theorem of Least Work provides a more suitable method for the analysis of partially or completely articulated structures. To illustrate the method, consider the three-span continuous truss (Fig. 2.21).

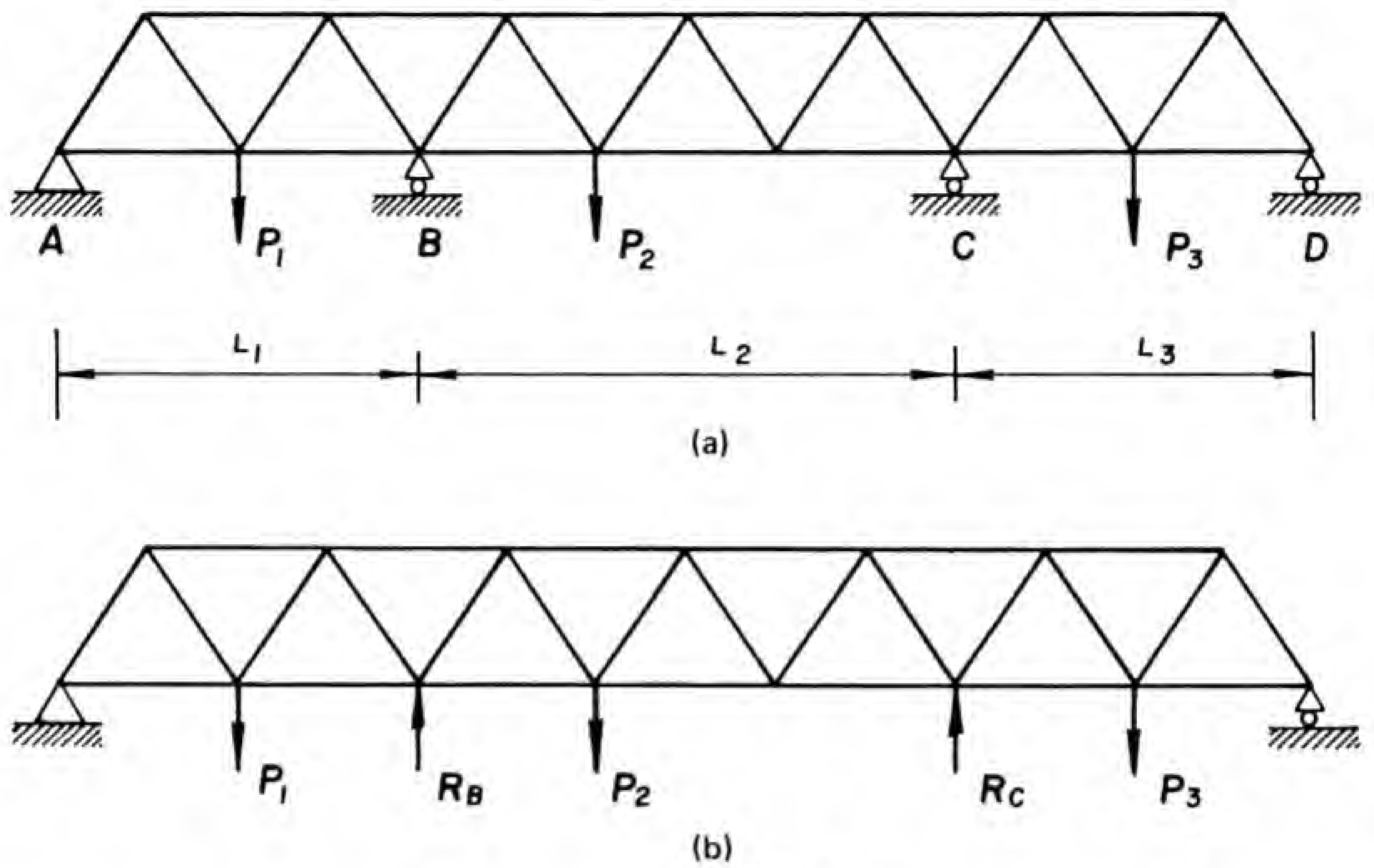


Figure 2.21

By removing the interior supports and replacing them by the forces R_B and R_C which are equal in magnitude and direction to the respective reactions, the structure of Fig. 2.21(b) becomes statically equivalent to the actual structure. The bar force in any member of the structure is

$$N = N_o + R_B n_B + R_C n_C \tag{2.35}$$

where

N_o = bar force in any member of the base structures due to applied loads

n_B = bar force in any member due to a unit vertical force applied at B

n_C = bar force in any member due to a unit vertical force applied at C

The expression for the total internal (strain) energy is

$$U = \frac{1}{2}\sum \frac{N^2 L}{EA} \tag{2.36}$$

The equations for determining the redundants are written from Castigliano's Theorem of Least Work:

$$\begin{aligned} \frac{\partial U}{\partial R_B} &= \frac{NL}{EA}\frac{\partial N}{\partial R_B} = 0 \\ \frac{\partial U}{\partial R_C} &= \frac{NL}{EA}\frac{\partial N}{\partial R_C} = 0 \end{aligned} \tag{2.37}$$

Note that

$$\frac{\partial U}{\partial R_B} = n_B$$

$$\frac{\partial U}{\partial R_C} = n_C$$

Substituting [2.37] into [2.35] gives

$$\begin{aligned} &\frac{N_o n_B L}{EA} + R_B \sum \frac{n_B^2 L}{EA} + R_C \sum \frac{n_B n_C L}{EA} = 0 \\ &\frac{N_o n_C L}{EA} + R_B \sum \frac{n_B n_C L}{EA} + R_C \sum \frac{n_C^2 L}{EA} = 0 \end{aligned} \qquad [2.38]$$

or, in terms of displacements,

$$\begin{aligned} &\Delta_b + R_B \delta_{bb} + R_C \delta_{bc} = 0 \\ &\Delta_c + R_B \delta_{cb} + R_C \delta_{cc} = 0 \end{aligned} \qquad [2.39]$$

These equations are identical in form with the equations obtained from the virtual work principle.

EXAMPLE 2.8 *Find the reaction at B and the bar force in member BF using Castigliano's Theorem of Least Work (Fig. 2.22). The cross-sectional area of the members, in cm², are shown in parentheses.*

The truss is indeterminate to the second degree with one redundant member (internal indeterminacy) and one redundant reaction (external indeterminacy). Taking member BF and the reaction at B as redundant, the two condition equations of least work are

$$\frac{\partial U}{\partial R_B} = \Sigma N \frac{\partial N}{\partial R_B} \frac{L}{A} = 0$$

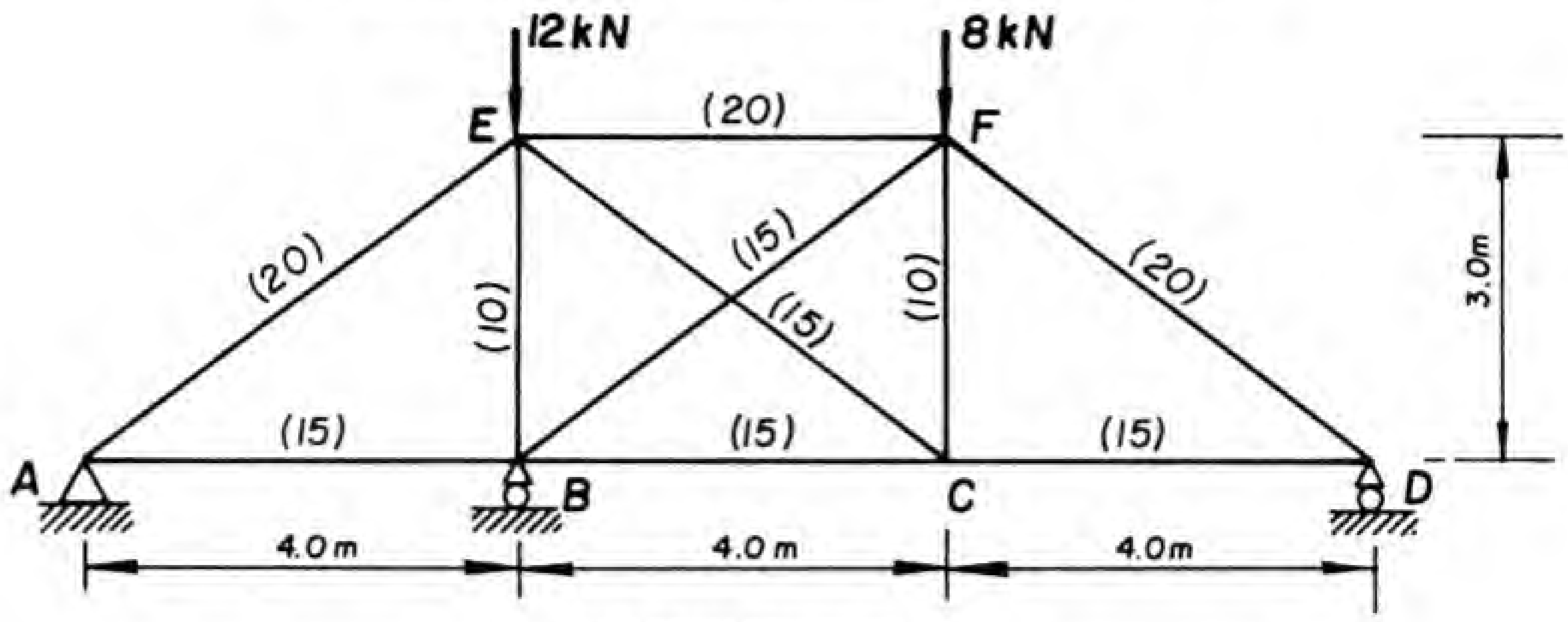
$$\frac{\partial U}{\partial F_{BF}} = \Sigma N \frac{\partial N}{\partial F_{BF}} \frac{L}{A} = 0$$

All computations necessary for the evaluation of the terms of the above equations are shown in Table 2.4.

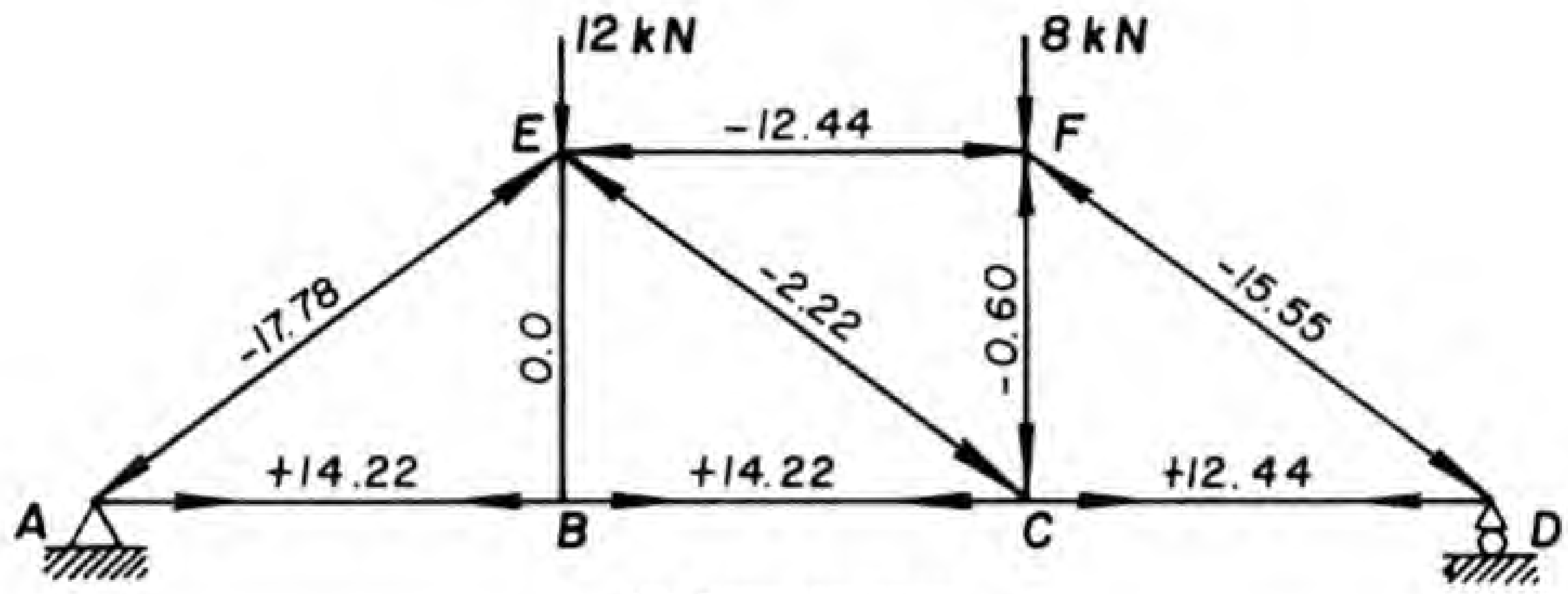
The required equations of least work are:

$$135.09 R_B + 54.53 F_{BF} - 1680.8 = 0$$

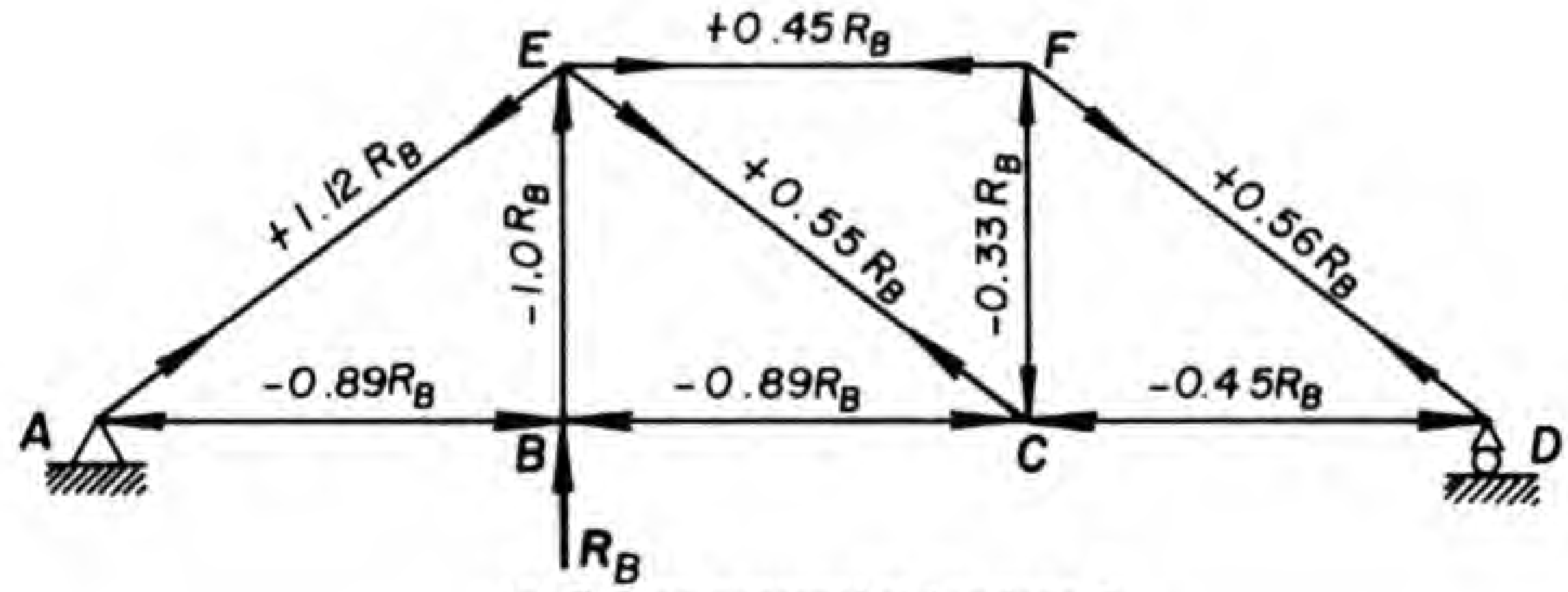
$$54.52 R_B + 118.2 F_{BF} - 202.3 = 0$$



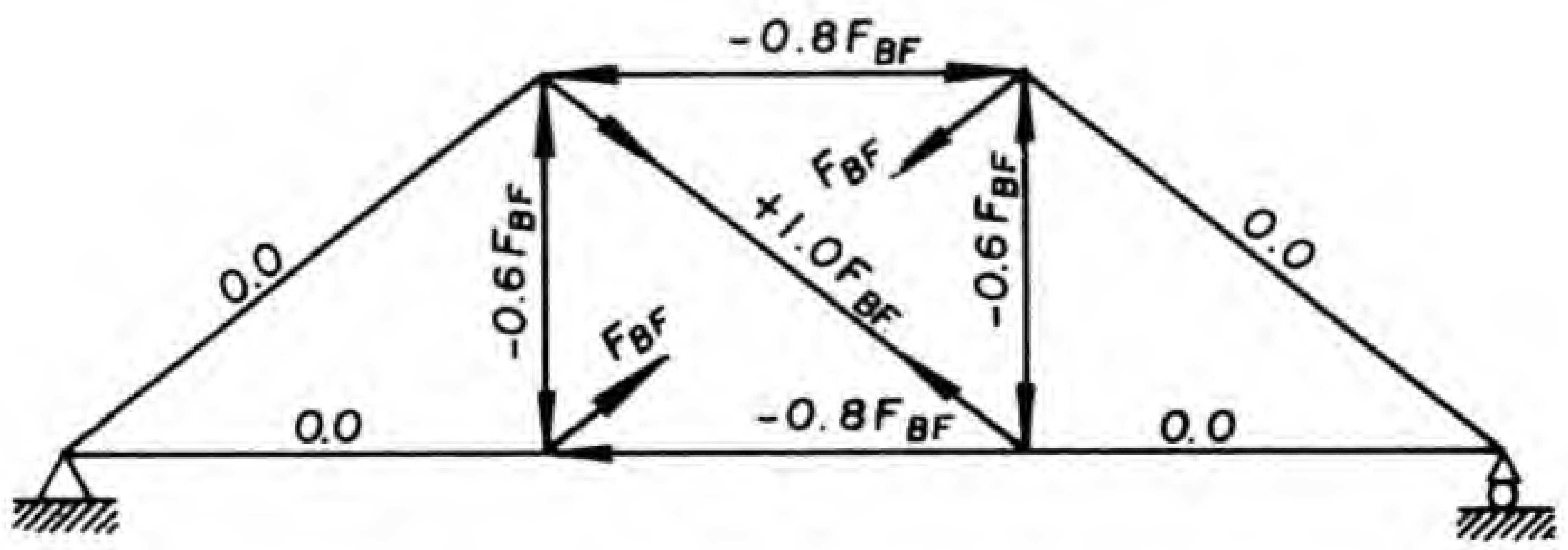
(a) Actual structure



(b) Base structure



(c) Due to external redundant R_B



(d) Due to internal redundant F_{BE}

Figure 2.22

Table 2.4

Member	L cm	A cm^2	$\frac{L}{A}$	Real	R_B	R_{BF}	$\frac{\partial N}{\partial R_B}$	$\frac{\partial N}{\partial F_{BE}}$	$N\frac{\partial N}{\partial R_B}\frac{L}{A}$	$N\frac{\partial N}{\partial F_{BF}}\frac{L}{A}$
AE	500	20	25	−17.78	$+1.12R_B$	0	1.12	0	$-497.84 + 31.36R_B + 0$	0
EF	400	20	20	−12.44	$+0.45R_B$	$-0.8F_{BF}$	0.45	−0.8	$-111.96 + 4.05R_B - 7.20F_{BF}$	$199.04 - 7.2R_B + 12.8F_{BF}$
DF	500	20	25	−15.55	$+0.55R_B$	0	0.55	0	$-213.81 + 7.56R_B + 0$	0
BE	300	10	30	0	$-1.00R_B$	$-0.6F_{BF}$	−1.00	−0.6	$0 + 30.0R_B + 18.0F_{BF}$	$0 + 18R_B + 10.8F_{BF}$
CE	500	15	33.33	+2.22	$+0.56R_B$	$+1.0F_{BF}$	0.56	1.0	$-41.44 + 10.45R_B + 18.67F_{BF}$	$-73.99 + 18.66R_B + 33.3F_{BF}$
BF	500	15	33.33	0	0	$+1.0F_{BF}$	0	1.0	0	$0 + 0 + 33.3F_{BF}$
CF	300	10	30	−1.33	$-0.33R_B$	$-0.6F_{BF}$	−0.33	−0.6	$13.17 + 3.27R_B + 5.94F_{BF}$	$-23.94R_B + 10.8F_{BF}$
AB	400	15	26.67	+14.22	$-0.90R_B$	0	−0.90	0	$-339.81 + 21.41R_B + 0$	0
BC	400	15	26.67	−14.22	$-0.90R_B$	$+0.8F_{BF}$	−0.90	−0.8	$-149.3 + 21.41R_B + 19.12$	$-303.40 + 19.12R_B + 17.1F_{BF}$
CD	400	15	26.67	−12.44	$-0.45R_B$	0	−0.45	0	$-149.3 + 5.40R_B + 0$	0
Σ									$-1680.8 + 135.09R_B + 54.53F_{BF}$	$-202.3 + 54.52R_B + 118.2F_{BF}$

The solution of the simultaneous equations is

R_B = 14.44 kN (upward)

F_{BF} = −4.95 kN (compression)

2.8 PROBLEMS

2.1 Evaluate the moments at supports A and B of the beam shown in Fig. P2.1.

(Ans: $M_A = -0.0626L^2$, $M_B = -0.00361L^2$)

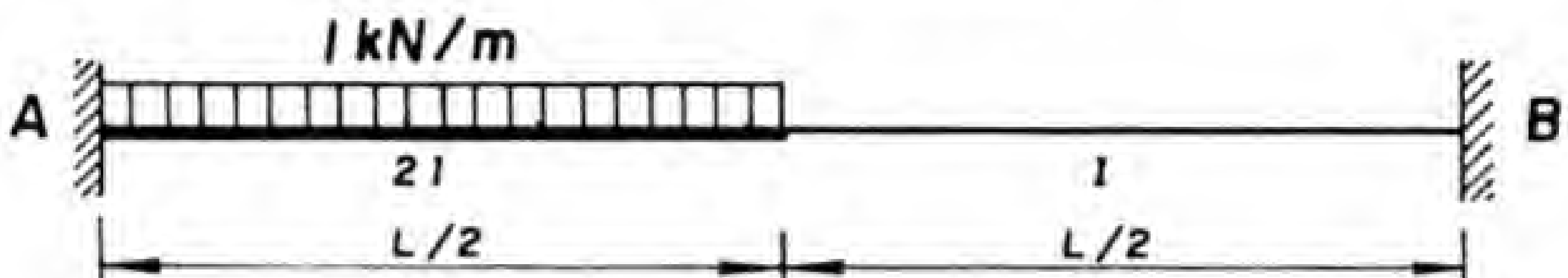


Figure P2.1

2.2 Determine the reactions and bar forces for the truss shown in Fig. P2.2.

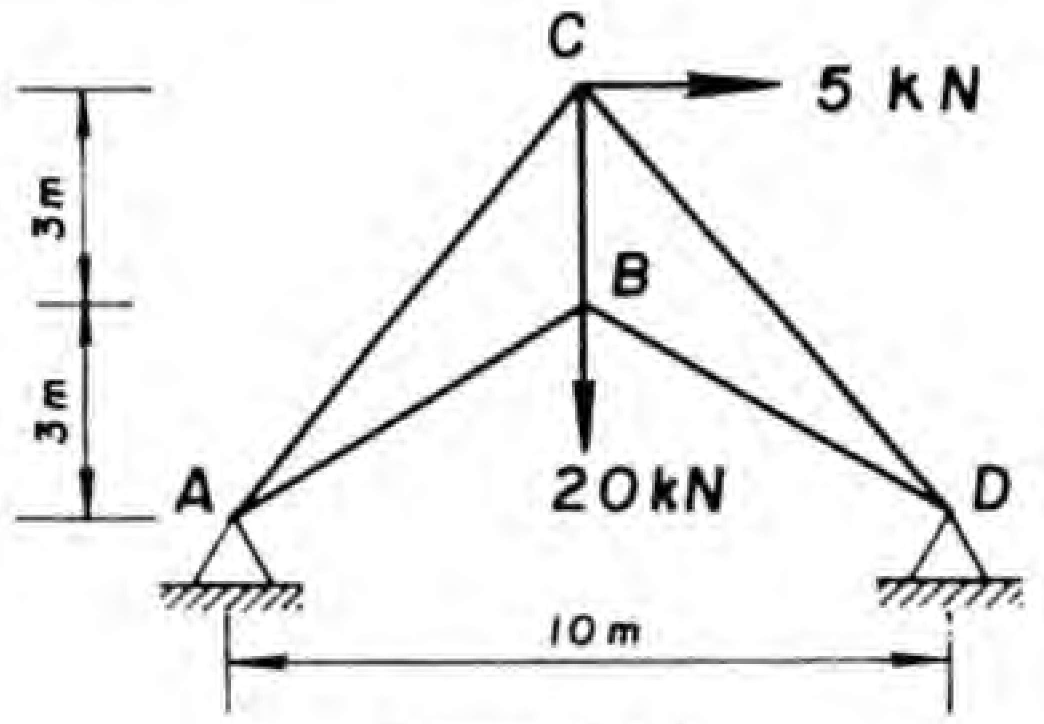


Figure P2.2

2.3 Determine the bar forces in the truss double diagonal system shown in Fig. P2.3. The area of all top chord members is $2A$ and the area of the remaining members is A.

(Ans: N_{DF} = −3.05 kN N_{BE} = −5.56 kN N_{BD} = −3.65 kN

N_{EF} = 3.00 kN N_{CF} = −2.80 kN N_{BF} = −3.76 kN

N_{AD} = −2.76 kN N_{AF} = 4.60 kN N_{CE} = 4.66 kN)

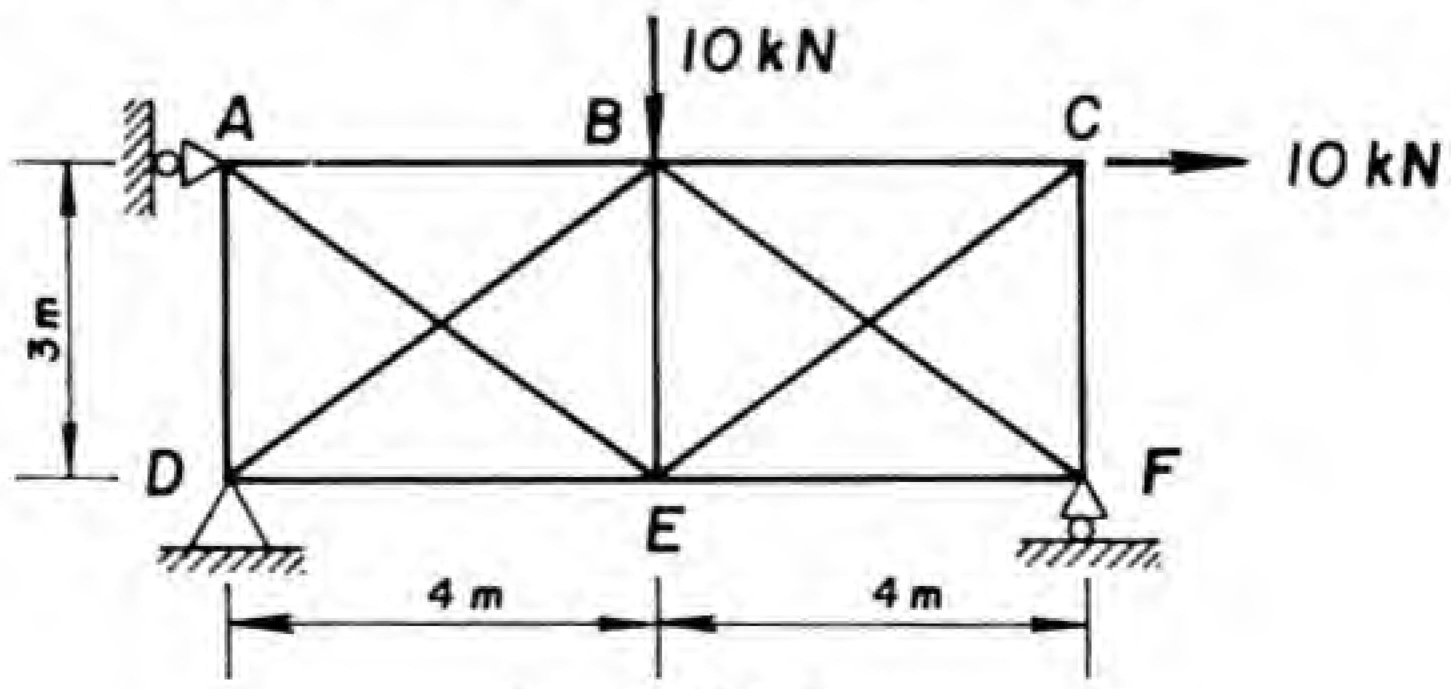


Figure P2.3

2.4 Construct the shear and bending moment for the continuous beam shown in Fig. P2.4.

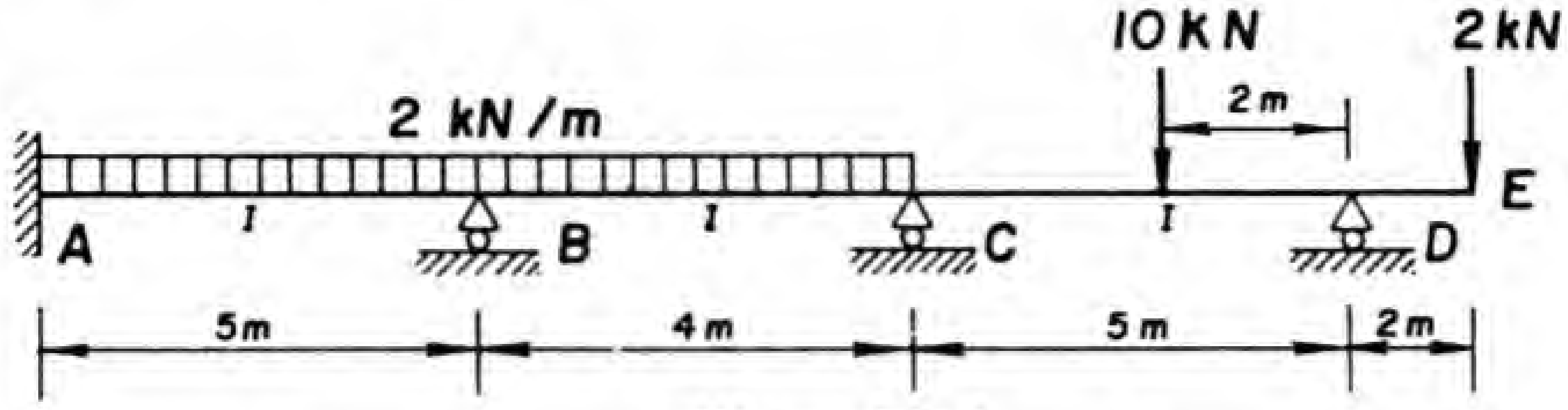


Figure P2.4

2.5 Draw the bending moment of the frame shown in Fig. P2.5.

(*Ans*: $M_A = +5.40$ kN m $\quad$ $M_D = -2.65$ kN m

$M_B = -6.75$ kN m $\quad$ $M_E = +2.71$ and -7.29 kN m

$M_C = +7.09$ kN m)

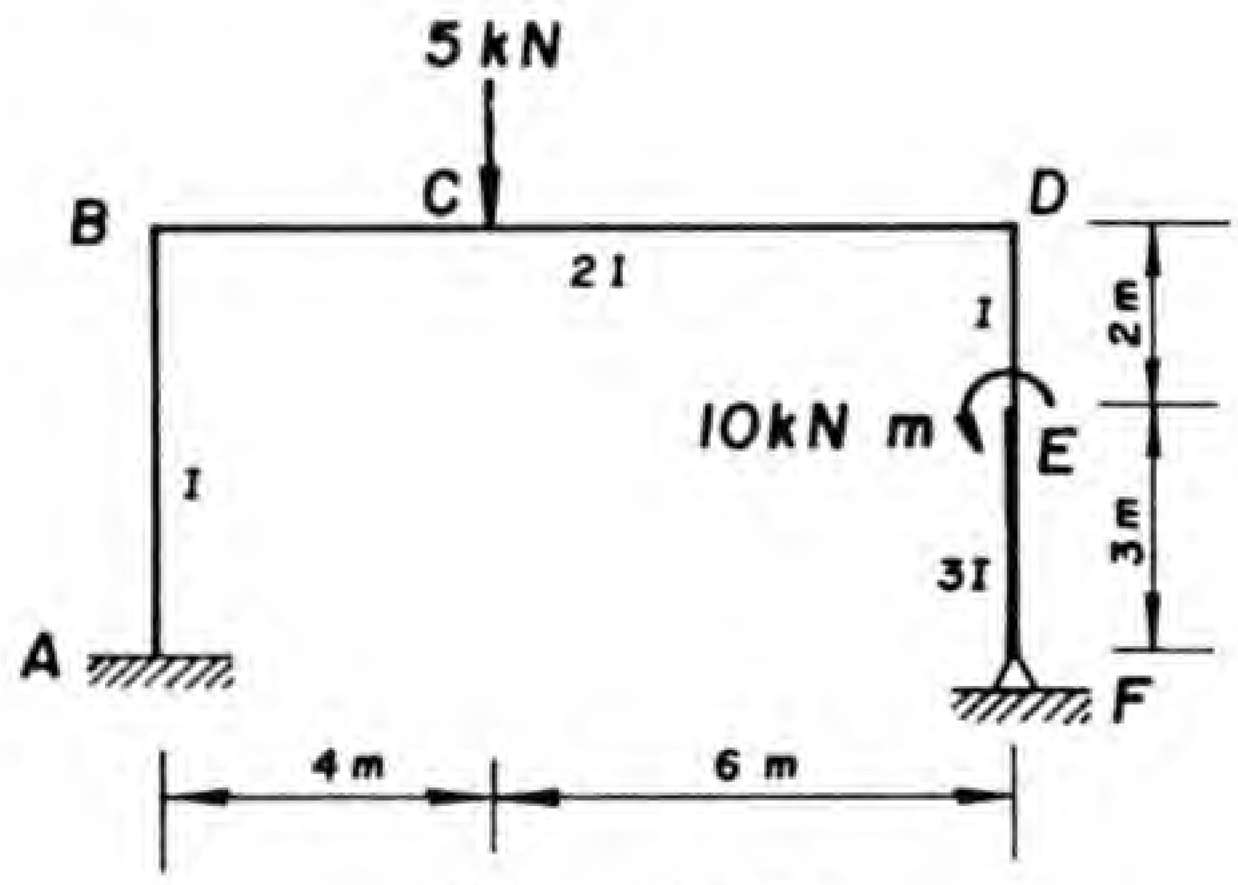


Figure P2.5

2.6 Find the support moments of the continuous beam in Fig. P2.4 using the method of three moments.

2.7 Find the joint moments of the frame shown in Fig. P2.6 using the elastic centre method.

(*Ans*: $M_B = -M_D = 0.42$ kN m)

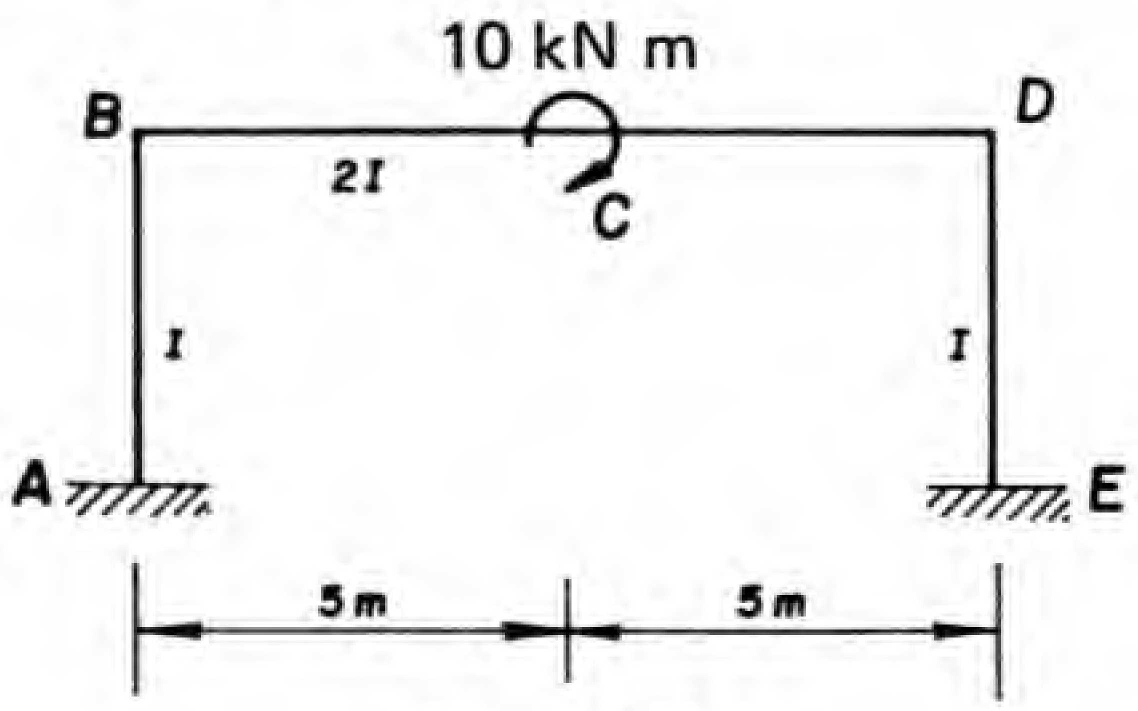


Figure P2.6

3. The Slope Deflection Method

3.1 INTRODUCTION

The slope deflection method was presented by G. A. Maney in 1915 as a general method of analysis for continuous beams and rigid-jointed frames. One of the main values of the slope deflection method lies in that it forms the basis of other methods such as the Cross and Kani moment distribution methods. These methods are numerical iteration solution of the slope deflection simultaneous equations.

In the slope deflection method the rotations and relative joint displacements are the unknowns and the moments at the joints are found in terms of rotations, joint displacements, stiffness and length of members which are determined from the solution of simultaneous algebraic equations. The method assumes all joints to be rigid; that is, the angles between the members at the joints do not change under applied loads. Also, displacements due to axial and shearing forces, being very small, are neglected. Consequently, the joint is considered to rotate as a whole and a single angle of rotation is sufficient to define the rotation of all members at a joint. By properly determining the joint rotation and displacements at the ends of the individual members, the joint moments are formulated from the equations of equilibrium.

3.2 DEVELOPMENT OF SLOPE DEFLECTION EQUATIONS

3.2.1 Sign Convention

There are different sign conventions used in different textbooks. In all subsequent discussion a ***statical sign convention*** suitable for beam and frame

analysis is adopted, as follows.

(a) Moments An end moment M is considered *positive* if it tends to rotate the *member counter-clockwise* or the *joint clockwise*. That is, if the member tends to rotate the support clockwise, the moment is considered clockwise.

(b) Joint Rotation The rotation θ of a joint is *positive* if the tangent turns in a *clockwise* direction.

(c) Member Rotation The rotation of the chord connecting the ends (Δ/L) is *positive* if the member turns in *clockwise* direction.

3.2.2 The Slope Deflection Equations

To develop the slope deflection equations, consider a base structure which is a straight prismatic restrained beam AB of span L and moment of inertia I subjected to a system of loads. Due to the effect of loads and the adjacent members, the beam is displaced as shown in Fig. 3.1(b). With reference to the original position, the ends rotate through θ_A and θ_B at A and B, respectively, and a relative vertical displacement of Δ of the member ends. The end moments produced at A and B are, respectively M_{AB} and M_{BA}. The member AB is equivalent, both statically and kinematically, to the sum of those shown in Fig. 3.1(d)–(f). The effect of each displacement component to determine the end moments at A and B is studied separately. Note that the end and member rotations are assumed clockwise and therefore positive.

These components are:

(a) Fixed-end moments

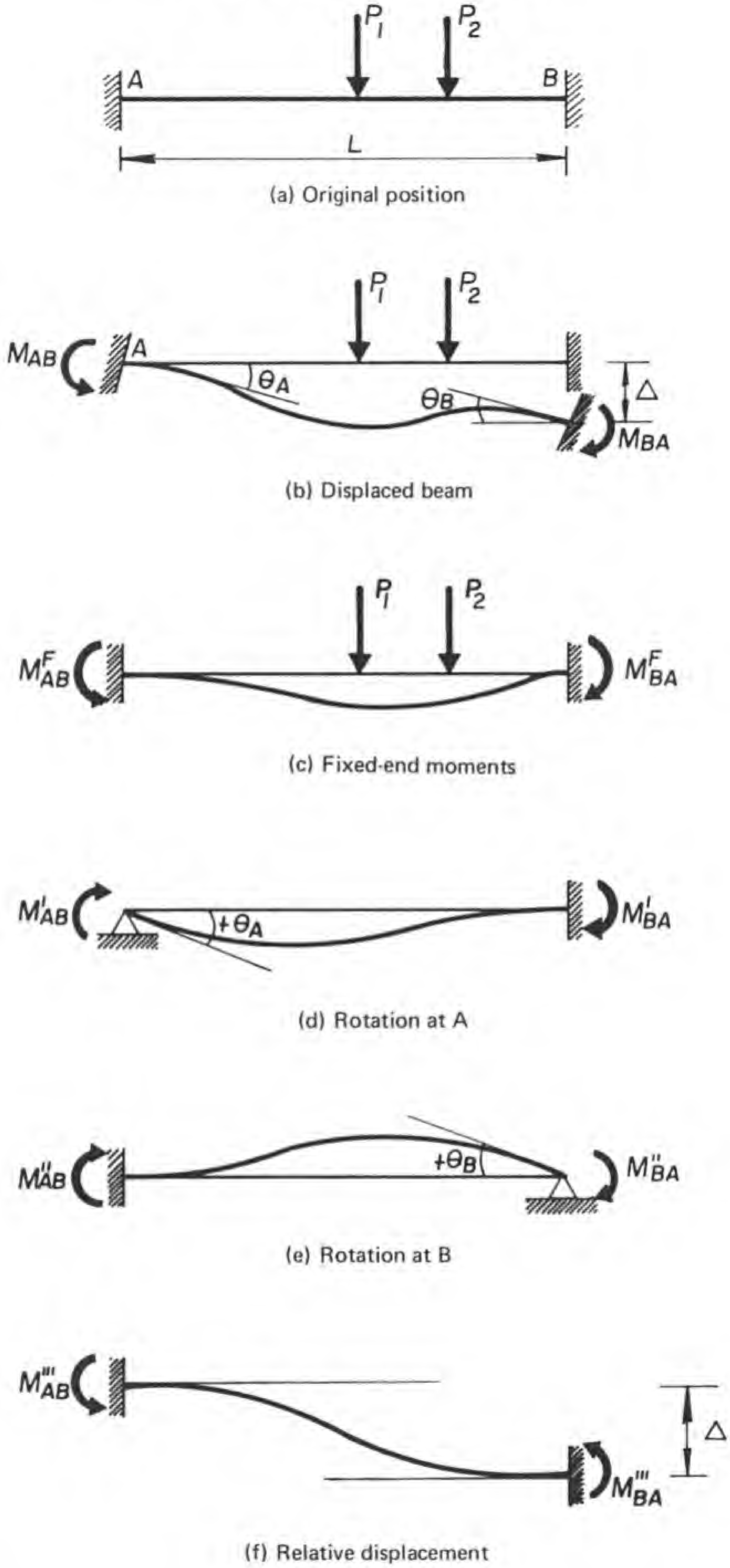
For the applied load system the fixed-end moments M_{AB}^F and M_{BA}^F are first determined.

(b) End moments due to rotation θ_A

The member AB is propped cantilever with clockwise rotation θ_A (Fig. 3.1(d)). Using the conjugate beam method, the rotation and displacement at support A are

$$\theta_A = \frac{(M'_{AB} + M'_{BA})L}{2EI}$$

$$\Delta_A = \frac{M'_{AB}L}{2EI}\left(\frac{L}{3}\right) + \frac{M'_{BA}L}{2EI}\left(\frac{2L}{3}\right) = 0$$



P1
P2
A
B
L
(a) Original position
P1
P2
MAB
A
θA
θB
Δ
MBA
(b) Displaced beam
P1
P2
M^F_AB
M^F_BA
(c) Fixed-end moments
M'AB
+θA
M'BA
(d) Rotation at A
M''AB
+θB
M''BA
(e) Rotation at B
M'''AB
Δ
M'''BA
(f) Relative displacement

Figure 3.1

From which

$$M'_{BA} = -\frac{1}{2}M'_{AB}$$

Therefore

$$\theta_A = \frac{M'_{AB}L}{4EI}$$

Hence

$$M'_{AB} = \frac{4EI\theta_A}{L} \quad \text{and} \quad M'_{BA} = -\frac{2EI\theta_A}{L} \qquad [3.1]$$

(c) End moments due to rotation θ_B (Fig. 3.1(e))

In a similar manner as above,

$$\theta_B = -\frac{(M''_{AB} + M''_{BA})L}{2EI}$$

and

$$\Delta_B = -\frac{M''_{BA}L}{2EI}\left(\frac{L}{3}\right) - \frac{M''_{AB}L}{2EI}\left(\frac{2L}{3}\right) = 0$$

From which

$$M''_{AB} = -\frac{1}{2}M''_{BA}$$

Therefore

$$\theta_B = -\frac{M''_{BA}L}{4EI}$$

Hence

$$M''_{BA} = -\frac{4EI\theta_B}{L} \quad \text{and} \quad M''_{AB} = \frac{2EI\theta_B}{L} \qquad [3.2]$$

(d) End moments due to relative displacement Δ (Fig. 3.1(f))

The slope at support A is

$$\theta_A = \frac{(M'''_{AB} + M'''_{BA})L}{2EI} = 0$$

Thus

$$M'''_{AB} = -M'''_{BA}$$

Using the conjugate beam method

$$\Delta = \frac{M'''_{BA}}{2EI}\left(\frac{L}{3}\right) + \frac{M'''_{AB}}{2EI}\left(\frac{2L}{3}\right)$$

$$= \frac{M'''_{BA}L^2}{6EI}$$

Hence

$$M'''_{AB} = -\frac{6EI\Delta}{L^2} \quad \text{and} \quad M'''_{BA} = +\frac{6EI\Delta}{L^2} \qquad [3.3]$$

The separate moments of Fig. 3.1(c)–(f) may be superimposed to represent the true end moments for beam AB according to the bending moment sign convention:

$$M_{AB} = M^F_{AB} + M'_{AB} + M''_{AB} + M'''_{AB}$$

$$M_{BA} = M^F_{BA} + M'_{BA} + M''_{BA} + M'''_{BA}$$

or in terms of the end rotations and displacements:

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B - 3\Delta/L)$$

$$M_{BA} = M^F_{BA} - \frac{2EI}{L}(2\theta_B + \theta_A - 3\Delta/L)$$

Adopting the statical sign convention and introducing $K = I/L$ the equations are revised as

$$\begin{aligned} M_{AB} &= M^F_{AB} - 2EK(2\theta_A + \theta_B - 3\Delta/L) \\ M_{BA} &= M^F_{BA} - 2EK(2\theta_B + \theta_A - 3\Delta/L) \end{aligned} \qquad [3.4]$$

The above equations are known as the *slope deflection equations*. They express the end moments of a member in terms of its end displacement. Note that the sign of the fixed-end moment is shown as plus. However, its correct sign will be determined by the direction it tends to rotate the joint following the adopted sign convention.

These equations can further be generalised as a single equation in the form

$$M_{jm} = M^F_{jm} - 2EK_{jm}(2\theta_j + \theta_m - 3\Delta_{jm}/L_{jm}) \qquad [3.5]$$

where the subscripts j and m represent the near and far end of member JM respectively.

The slope deflection equations given by [3.5] represent two equations with

six unknowns. The unknowns are solved by applying boundary conditions for the displacements and continuity conditions for the moments.

3.2.3 Fixed-End Moments

When a beam whose supports are completely fixed against rotation or translation is subjected to transverse loads the beam is called a *fixed-end beam*. The bending moments at the supports of such a beam are called *fixed-end moments*. The values of the fixed-end moments for common types of loadings are given in Table 3.1.

3.3 APPLICATION OF SLOPE DEFLECTION EQUATIONS TO BEAM PROBLEMS

3.3.1 Beams With No Support Settlements

The slope deflection equations for a member with no relative lateral displacement between the ends are:

$$\begin{aligned} M_{AB} &= M^F_{AB} - 2EK(2\theta_A + \theta_B) \\ M_{BA} &= M^F_{BA} - 2EK(2\theta_B + \theta_A) \end{aligned} \qquad [3.6]$$

Consider the two-span beam shown in Fig. 3.2.

Fixed-end Moments

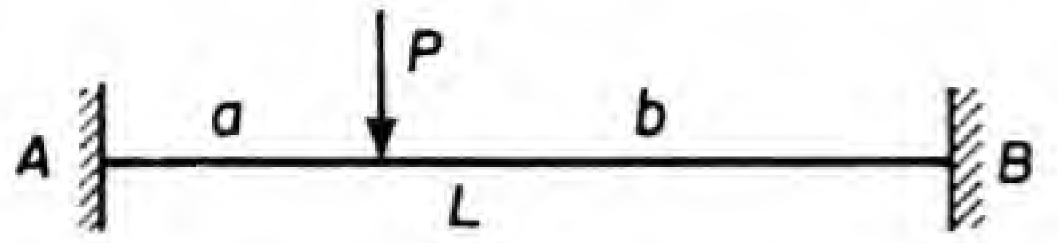
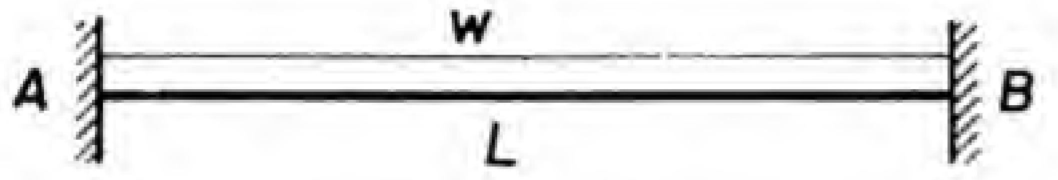
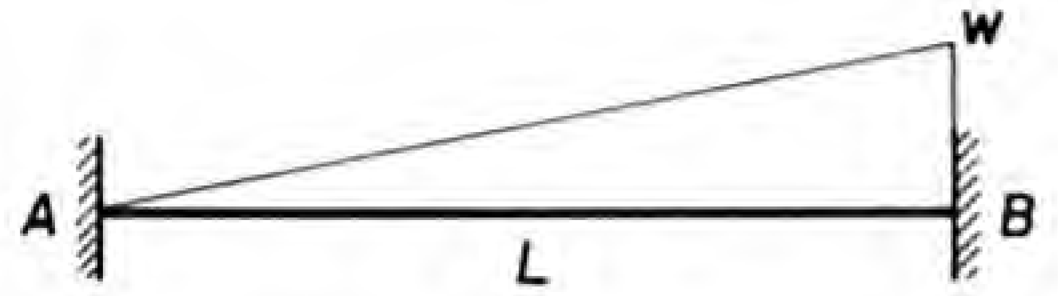
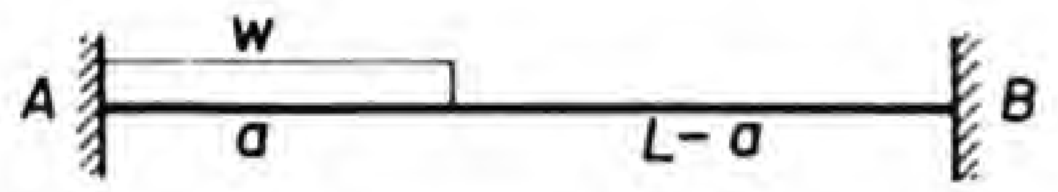
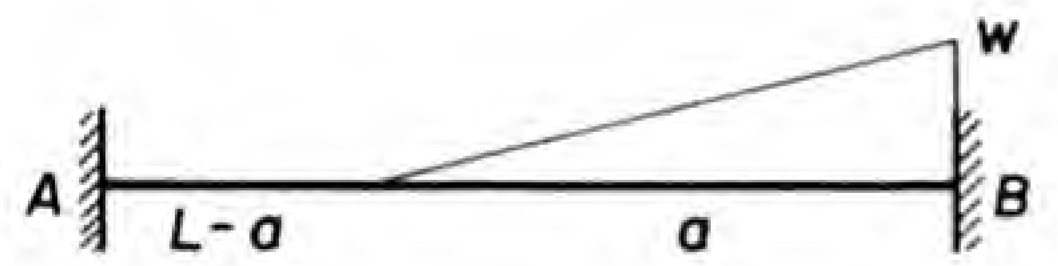
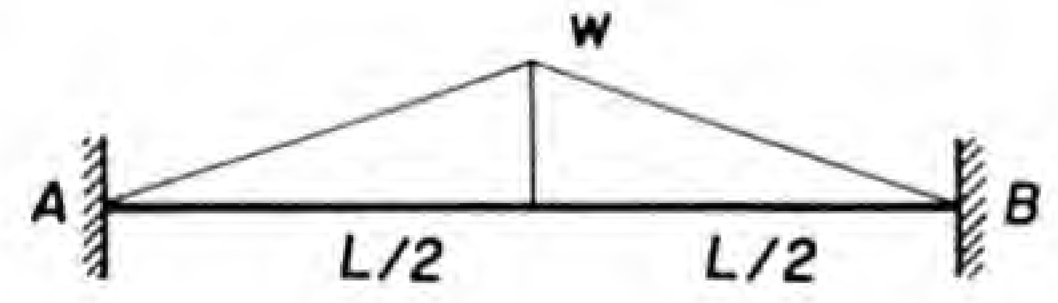
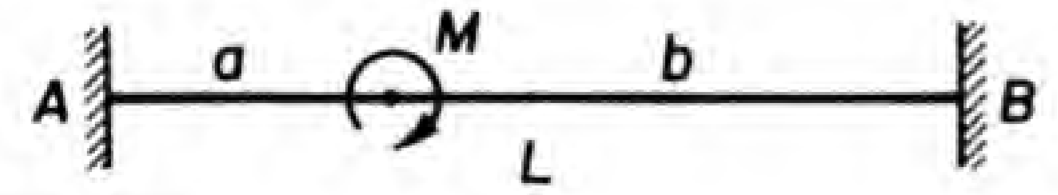
The fixed-end moments depend on the applied loading. The fixed-end moments are M^F_{AB} and M^F_{BA} for span AB and M^F_{BC} and M^F_{CB} for span BC.

Slope Deflection Equations

The slope deflection equations are written for each member as follows:

$$\begin{aligned} M_{AB} &= M^F_{AB} + 2EK_{AB}(2\theta_A + \theta_B) \\ M_{BA} &= M^F_{BA} + 2EK_{AB}(\theta_A + 2\theta_B) \\ M_{BC} &= M^F_{BC} + 2EK_{BC}(2\theta_B + \theta_C) \\ M_{CB} &= M^F_{CB} + 2EK_{BC}(\theta_B + 2\theta_C) \end{aligned} \qquad [3.7]$$

Table 3.1

	M^F_{AB}	M^F_{BA}
	$\frac{Pab^2}{L^2}$	$-\frac{Pa^2b}{L^2}$
	$\frac{wL^2}{12}$	$-\frac{wL^2}{12}$
	$\frac{wL^2}{30}$	$-\frac{wL^2}{20}$
	$\frac{wL^2}{12}\left(6-8\frac{a}{L}+3\frac{a^2}{L^2}\right)$	$-\frac{wL^2}{12}\left(4-3\frac{a}{L}\right)$
	$\frac{wa^3}{60L}\left(5-3\frac{a}{L}\right)$	$-\frac{wa^2}{60}\left(16-10\frac{a}{L}+3\frac{a^2}{L^2}\right)$
	$\frac{5wL^2}{96}$	$-\frac{5wL^2}{96}$
	$-b(2a-b)\frac{M}{L^2}$	$-a(2b-a)\frac{M}{L^2}$

Equilibrium Condition of Joints

The equilibrium equations are written for each joint by taking the free-body diagram of the joints. These equations are

$$\Sigma M_A = M_{AB} = 0$$

$$\Sigma M_B = M_{BA} + M_{BC} = 0 \qquad [3.8]$$

$$\Sigma M_C = M_{CB} = 0$$

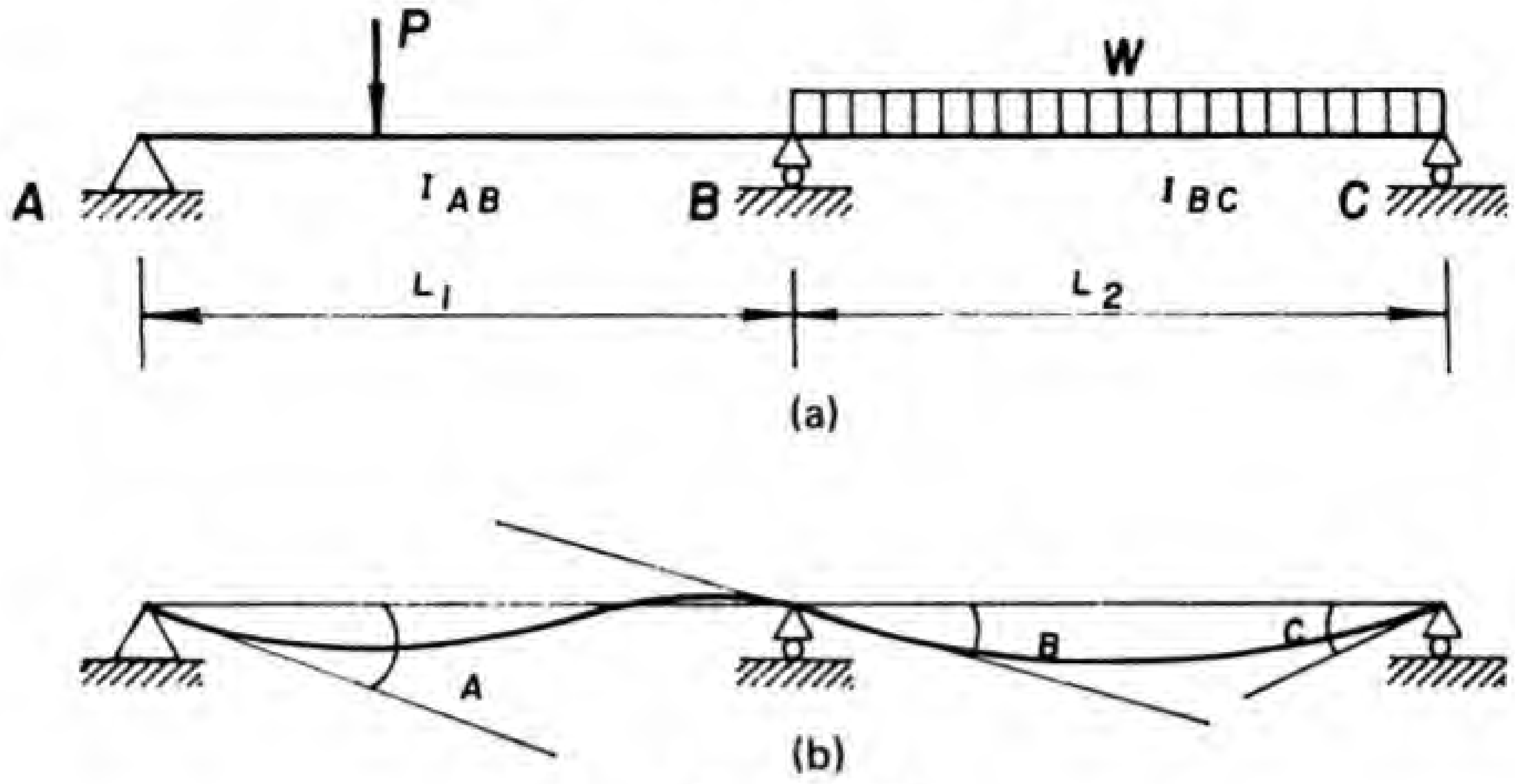


Figure 3.2

The equilibrium equations [3.8] may be written in terms of the rotations given in [3.7]. Hence

$$M_{AB}^F + 4EK_{AB}\theta_A + 2EK_{AB}\theta_B = 0$$

$$M_{BA}^F + M_{BC}^F + 2EK_{AB}\theta_A + (4EK_{AB} + 4EK_{BC})\theta_B + 2EK_{BC}\theta_C = 0 \qquad [3.9]$$

$$M_{CB}^F + 2EK_{BC}\theta_B + 4EK_{BC}\theta_C = 0$$

The above equations may be written in matrix notation as follows:

$$\begin{bmatrix} M_{AB}^F \\ M_{BA}^F + M_{BC}^F \\ M_{CB}^F \end{bmatrix} + \begin{bmatrix} 4EK_{AB} & 2EK_{AB} & 0 \\ 2EK_{AB} & 4E(K_{AB} + K_{BC}) & 2EK_{BC} \\ 0 & 2EK_{BC} & 2EK_{BC} \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad [3.10]$$

The above equation may be written in compact form as

$$\{M^F\} + [K]\{\theta\} = \{0\} \qquad [3.11]$$

where $\{M^F\}$ is the vector of the fixed-end moments, $[K]$ is the member stiffness matrix, and $\{\theta\}$ is the vector of end rotations. In [3.10] the unknown quantities are the end rotations which can be determined from the solution of the simultaneous equations. These rotation quantities can then be substituted into [3.7] to find the support moments.

The procedure for analysing continuous beams is as follows:

(a) Determine the fixed-end moments in each span.
(b) Use the slope deflection equations to express the end-moments and end-rotations.
(c) Establish the equilibrium equations of moments where the support rotations are the unknowns, at each joint capable of rotating: the sum of the end moments of all members at the joint is zero.

(d) Evaluate the rotations by solving the simultaneous equations.
(e) Substitute the rotations back into the slope deflection equations to compute the end moments.

EXAMPLE 3.1 *Determine the support moments for the beam shown in Fig. 3.3.*

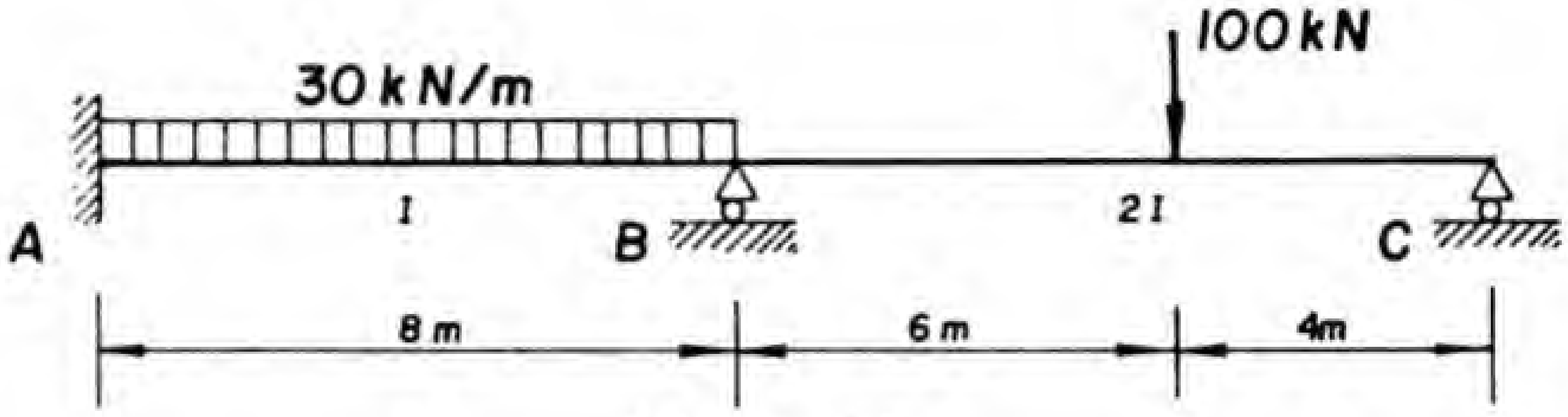


Figure 3.3

Relative Stiffness

$$(K_{AB}) : (K_{BC}) = \left(\frac{I}{8}\right) : \left(\frac{2I}{10}\right) = 5{:}8$$

Fixed-end Moments

$$M^F_{AB} = -M^F_{BA} = \frac{30 \times 8^2}{12} = 160 \text{ kN m}$$

$$M^F_{BC} = \frac{100 \times 6 \times 4^2}{10^2} = 96 \text{ kN m}$$

$$M^F_{CB} = -\frac{100 \times 4 \times 6^2}{10^2} = -144 \text{ kN m}$$

Slope Deflection Equations

$$M_{AB} = 160 - 5(2\theta_A + \theta_B)$$

$$= 160 - 5\theta_B \qquad \text{(since } \theta_A = 0)$$

$$M_{BA} = -160 - 5(2\theta_B)$$

$$M_{BC} = 96 - 8(2\theta_B + \theta_C)$$

$$M_{CB} = -144 - 8(\theta_B + 2\theta_C)$$

Equilibrium Equations

At joint B : $M_{BA} + M_{BC} = 0$

At joint C : $M_{CB} = 0$

Hence

$$26\theta_B + 8\theta_C = -64$$

$$8\theta_B + 16\theta_C = -144$$

In matrix form

$$\begin{bmatrix} 26 & 8 \\ 8 & 16 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} -64 \\ -144 \end{bmatrix}$$

Solution of the above equation yields

$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0.364 \\ -9.182 \end{bmatrix}$$

End Moments

Substitution of θ values into the slope deflection equations yields the support moments according to the statical sign convention:

$$M_{AB} = 160 - 5(0.364) = 158.18 \text{ kN m}$$

$$M_{BA} = -160 - 5(0.726) = -163.64 \text{ kN m}$$

$$M_{BC} = 96 - 8(0.728 - 9.182) = +163.64 \text{ kN m}$$

$$M_{CB} = -144 - 8(0.364 - 18.354) = 0.0 \text{ kN m}$$

3.3.2 Beams With Support Settlements

Support yielding in continuous beams induces bending moments at all supports of the beam. The general slope deflection equations are used to analyse continuous beams. The equations are

$$M_{AB} = M_{AB}^F - 2EK(2\theta_A + \theta_B - 3\Delta/L)$$

$$M_{BA} = M_{BA}^F - 2EK(2\theta_B + \theta_A - 3\Delta/L) \qquad [3.4]$$

Usually the effect of support yielding only is investigated, and the result may then be combined with those of applied loadings.

EXAMPLE 3.2 *Determine the support moments of the continuous beam shown in Fig. 3.4. The support at A rotates through 0.15 radian in a clockwise direction and the support at C settles down 10 mm; $E = 2.1 \times 10^5$ N/mm^2 and $I = 4 \times 10^4$ mm^4*

$$EI = 2.1 \times 10^5 \times 4000 \times 10^{-6} = 8400 \text{ kN/m}^2$$

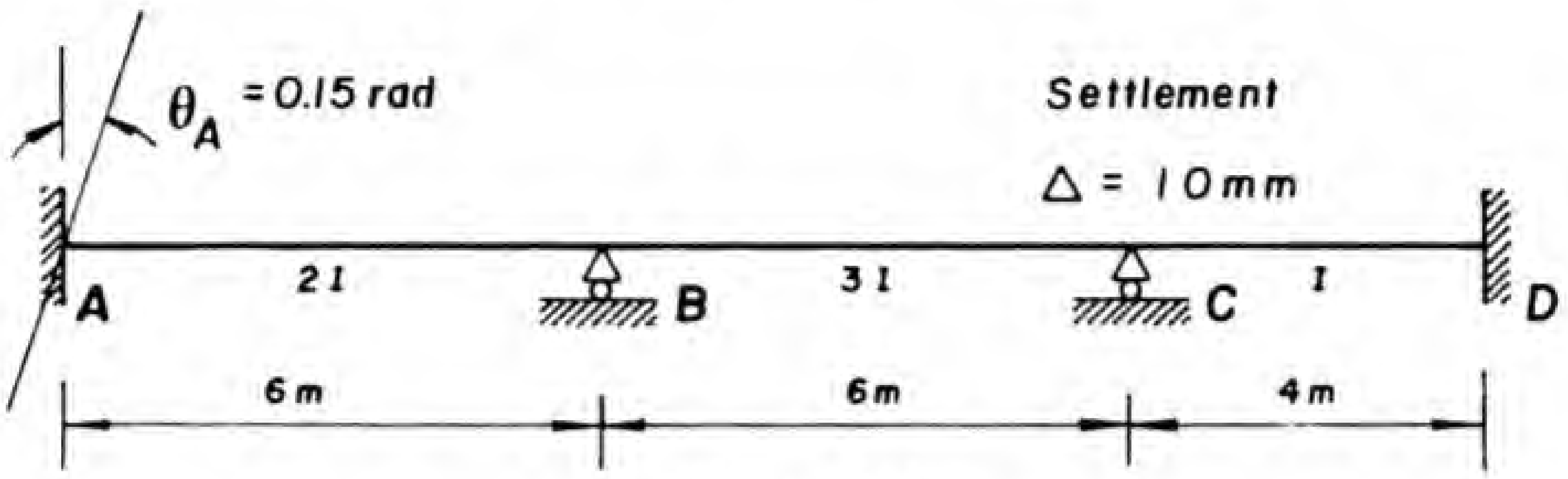


Figure 3.4

Slope Deflection Equations

$$M_{AB} = -\frac{2 \times 8400 \times 2}{6}(2\theta_A + \theta_B), \text{ where } \theta_A = +0.15 \text{ radian}$$

$$= -1680 - 5600\theta_B$$

$$M_{BA} = -\frac{2 \times 8400 \times 2}{6}(2\theta_B + 0.15)$$

$$= -1120\theta_B - 840$$

$$M_{BC} = -\frac{2 \times 8400 \times 3}{6}(2\theta_B + \theta_C - 3\Delta/6) \quad (\text{where } \Delta = 10 \text{ mm} = 0.01 \text{ m})$$

$$= -16800\theta_B - 8400\theta_C + 42$$

$$M_{CB} = -\frac{2 \times 8400 \times 3}{6}\left(2\theta_C + \theta_B - \frac{3 \times 0.01}{6}\right)$$

$$= -1680\theta_C - 8400\theta_B + 42$$

$$M_{CD} = -\frac{2 \times 8400}{4}\left[2\theta_C + \theta_D - \left(-\frac{3 \times 0.01}{4}\right)\right] \quad (\text{where } \theta_D = 0)$$

$$= -8400\theta_C - 31.5$$

$$M_{DC} = -\frac{2 \times 8400}{4}\left(\theta_C + \frac{0.03}{4}\right)$$

$$= -4200\theta_C - 31.5$$

Joint Equilibrium Conditions

(i) Joint B : $M_{BA} + M_{BC} = 0$

(ii) Joint C : $M_{CB} + M_{CD} = 0$

Substituting the moment expressions into the equilibrium equations gives

$$\begin{bmatrix} -28000 & -8400 \\ -8400 & -25200 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 798 \\ -1050 \end{bmatrix}$$

Solving the simultaneous equations,

$\theta_B = -0.03181$ radian

$\theta_C = +0.01101$ radian

End Moments

$M_{AB} = -1680 - 5600\,(-0.03181) = -1501.9$ kN m

$M_{BA} = -11200\,(-0.03181) - 840 = -483.8$ kN m

$M_{BC} = -16800\,(-0.03181) - 8400\,(0.01101) + 42 = 483.9$ kN m

$M_{CB} = -16800\,(0.01101) - 8400\,(-0.03181) + 42 = +124.2$ kN m

$M_{CD} = -8400\,(0.01101) - 31.5 = -124.0$ kN m

$M_{DC} = -4200\,(0.01101) - 31.5 = -77.7$ kN m

Converting the statical sign convention into bending moment convention yields

$M_A = +1501.9$ kN m (tension at bottom fibre)

$M_B = -483.9$ kN m

$M_C = +124.1$ kN m

$M_D = 77.7$ kN m

3.4 APPLICATION OF SLOPE DEFLECTION EQUATIONS TO FRAMES

3.4.1 Frames Without Sidesway

The slope deflection equations for frames without sidesway are:

$$\begin{aligned} M_{AB} &= M^F_{AB} - 2EK(2\theta_A + \theta_B) \\ M_{BA} &= M^F_{BA} - 2EK(2\theta_B + \theta_A) \end{aligned} \qquad [3.6]$$

Under this case all joints of the frame remain fixed in location during loading. Such frames are either physically held against sidesway or are symmetric and subject to symmetric loading system.

In the slope deflection equations the joint rotations are considered as the unknown while the joint moments are determined from joint conditions of equilibrium. In effect, the slope deflection solution of a frame without sidesway is essentially the same as that of a continuous beam. For frames without sidesway, there are always as many conditions of equilibrium of joints as unknown rotations. After the rotations are determined, the joint moments can be found from the slope deflection equations.

3.4.2 Frames With Sidesway

When loads are applied to frames, there are cases in which lateral movement occur through unknown distances, although usually in known direction. The effect of such translation is to cause the joints to deflect relative to initial unloaded positions. In such cases, the joint rotations and relative displacements are the unknowns in the slope deflection equations. The governing slope deflection equations for members subjected to a sidesway are

$$\begin{aligned} M_{AB} &= M_{AB}^F - 2EK(2\theta_A + \theta_B - 3\Delta/L) \\ M_{BA} &= M_{BA}^F - 2EK(2\theta_B + \theta_A - 3\Delta/L) \end{aligned} \qquad [3.4]$$

Assuming that all axial deformations are so small as to cause no change in member lengths, the relative sidesway of the joints may be evaluated depending on the geometry of the frame.

Figure 3.5 shows a two-column frame subjected to a lateral force P applied at B. The frame will deflect to the right while joints B and C rotate clockwise. Thus, the frame has three unknown displacements namely θ_B, θ_C and Δ.

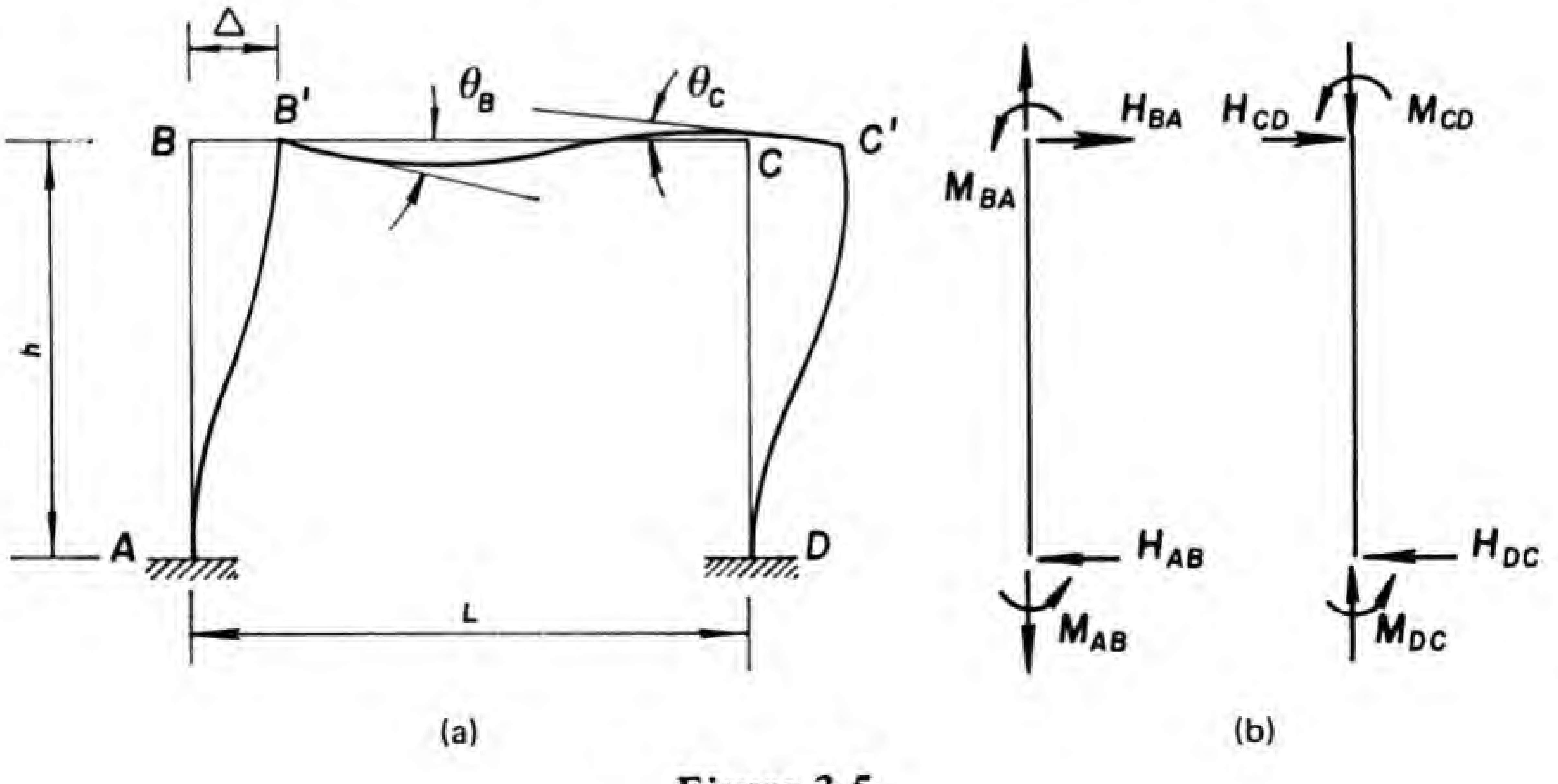


Figure 3.5

With three unknown displacements, three condition equations based on static equilibrium are required. Two equilibrium equations can be written for the two joints B and C. The third independent equation can be formulated by considering the horizontal static equilibrium of the frame. Taking the columns as free bodies, as in Fig. 3.5(b),

$$H_{AB} = \frac{M_{AB} + M_{BA}}{h}$$
$$H_{DC} = \frac{M_{CD} + M_{DC}}{h} \qquad [3.12]$$

Applying the equilibrium condition $\Sigma H = 0$ for the whole frame,

$$P - H_{AB} - H_{DC} = 0 \qquad [3.13]$$

This additional equation is generally known as the *sway equation* and sometimes known as the *shear equation* or the *bent equation*.

EXAMPLE 3.3 *Find the joint moments using the slope deflection equations of the frame shown in Fig. 3.6*

Relative Stiffness

Member	AB	BC	CD
Moment of inertia	I	$2I$	I
Span (m)	5	3	3
I/L	$I/5$	$2I/3$	$I/3$
Relative K	3	10	5

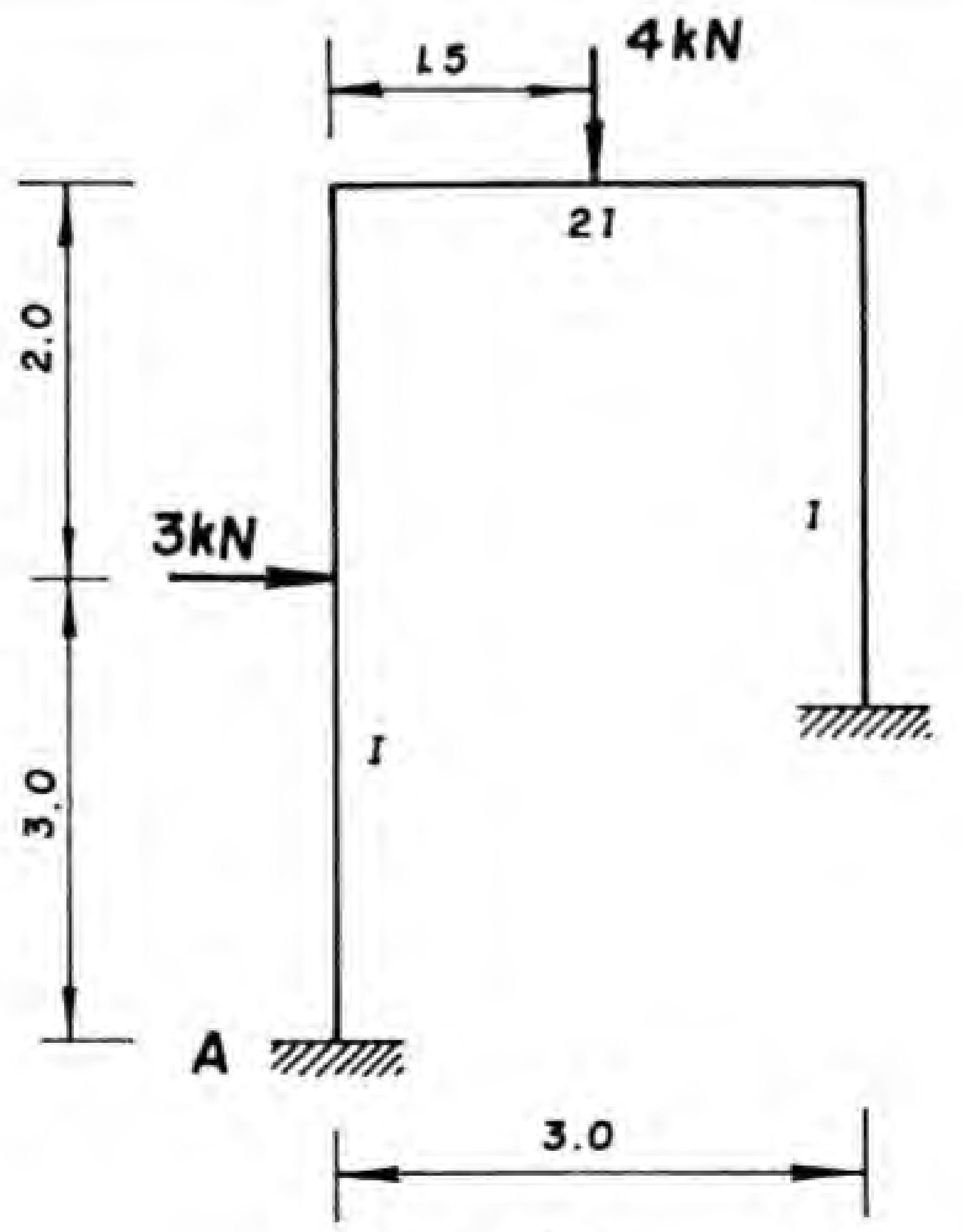


Figure 3.6

Fixed-end Moments

$$M_{AB}^F = \frac{3(3)(2)^2}{(5)^2} = +1.44 \text{ kN m}$$

$$M_{BA}^F = -\frac{3(2)(3)^2}{25} = -2.16 \text{ kN m}$$

$$M_{BC}^F = -M_{CB}^F = \frac{4(1.5)^3}{(3)^2} = 1.5 \text{ kN m}$$

Slope Deflection Equations

$$M_{AB} = 1.44 - 6(\theta_B - 0.6\Delta)$$
$$M_{BA} = -2.16 - 6(2\theta_B - 0.6\Delta)$$
$$M_{BC} = 1.5 - 20(2\theta_B + \theta_C)$$
$$M_{CB} = -1.5 - 20(\theta_B + 2\theta_C)$$
$$M_{CD} = -10(2\theta_C - \Delta)$$
$$M_{DC} = -10(\theta_C - \Delta)$$

Joint Equilibrium Equations

(i) At joint B : $M_{BA} + M_{BC} = 0$

(ii) At joint C : $M_{CB} + M_{CD} = 0$

Shear Condition

Substituting

$$H_A = \frac{M_{AB} + M_{BA} + 6}{5}$$

and

$$H_D = \frac{M_{CD} + M_{DC}}{3}$$

into the shear condition $3.0 - H_A - H_D = 0$:

$$3.0 - \left(\frac{M_{AB} + M_{BA} + 6}{5}\right) - \left(\frac{M_{CD} + M_{DC}}{3}\right) = 0$$

$$45 - 18 - 3(M_{AB} + M_{BA}) - 5(M_{CD} + M_{DC}) = 0$$

$$54\theta_B + 150\theta_C - 121.6\Delta = -29.16$$

or

$$-3.6\theta_B - 10.0\theta_C + 8.11\Delta = 1.944$$

Substituting the moment expressions into the joint equilibrium equations gives the following simultaneous equations:

$$\begin{bmatrix} 56 & 20 & -3.6 \\ 20 & 60 & -10 \\ -3.6 & -10 & 8.11 \end{bmatrix} \begin{bmatrix} \theta_B \\ \theta_C \\ \Delta \end{bmatrix} = \begin{bmatrix} -0.667 \\ -1.500 \\ 1.944 \end{bmatrix}$$

Solving the three simultaneous equations:

$$\theta_B = -0.00195$$

$$\theta_C = 0.0195$$

$$\Delta = 0.263$$

End moments

$$M_{AB} = 1.44 - 6[(-0.00195 - 0.6(0.263)] = 2.40 \text{ kN m}$$

$$M_{BA} = -2.16 - 6[2(-0.00195) - 0.6(0.263)] = -1.19 \text{ kN m}$$

$$M_{BC} = 1.5 - 20[2(-0.00195) + 0.0195] = +1.19 \text{ kN m}$$

$$M_{CB} = -1.5 - 20[-0.00195 + 2(0.0195)] = -2.24 \text{ kN m}$$

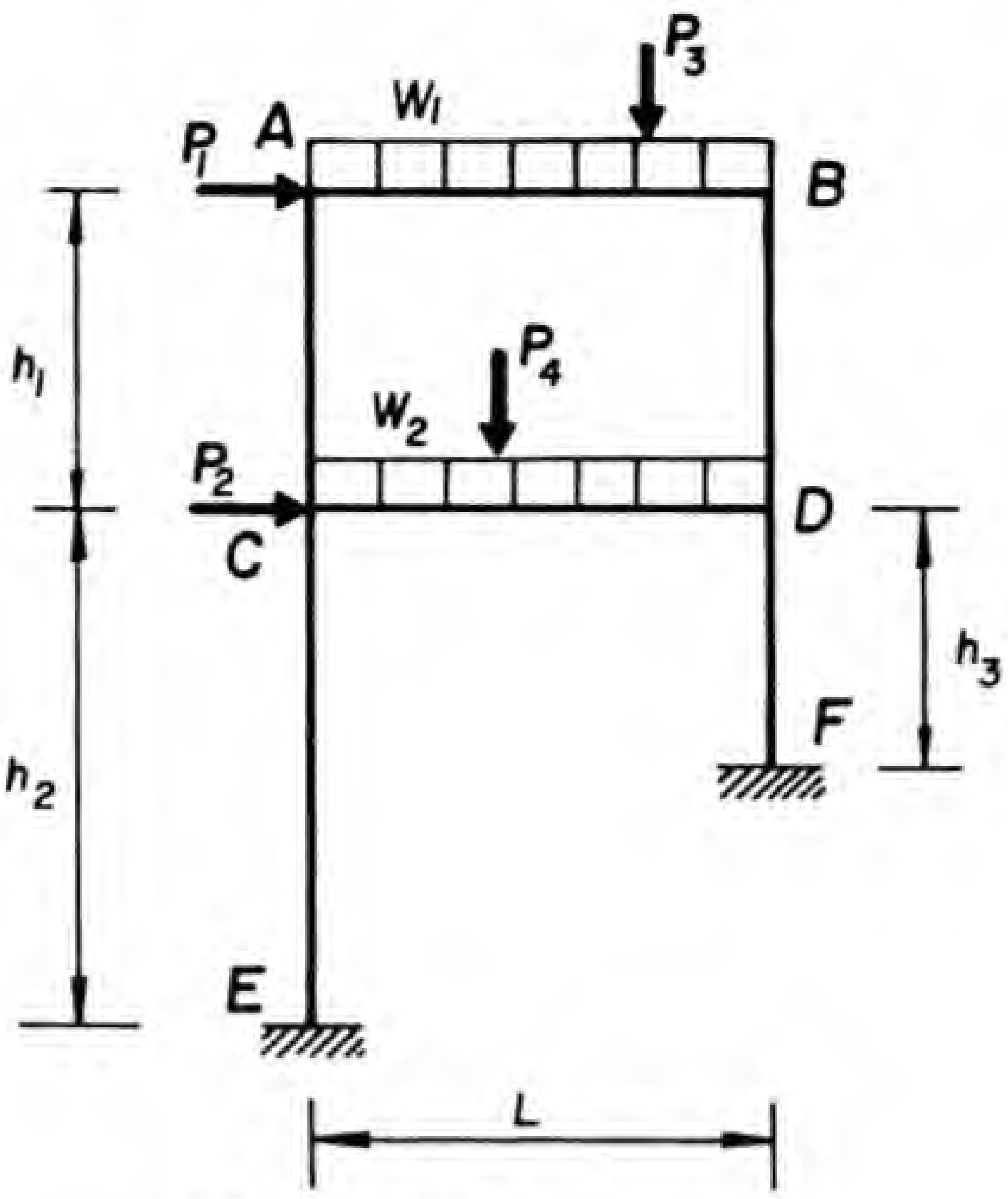
$$M_{CD} = -10[2(0.0195) - 0.263] = 2.24 \text{ kN m}$$

$$M_{DC} = -10(0.0195 - 0.263) = 2.44 \text{ kN m}$$

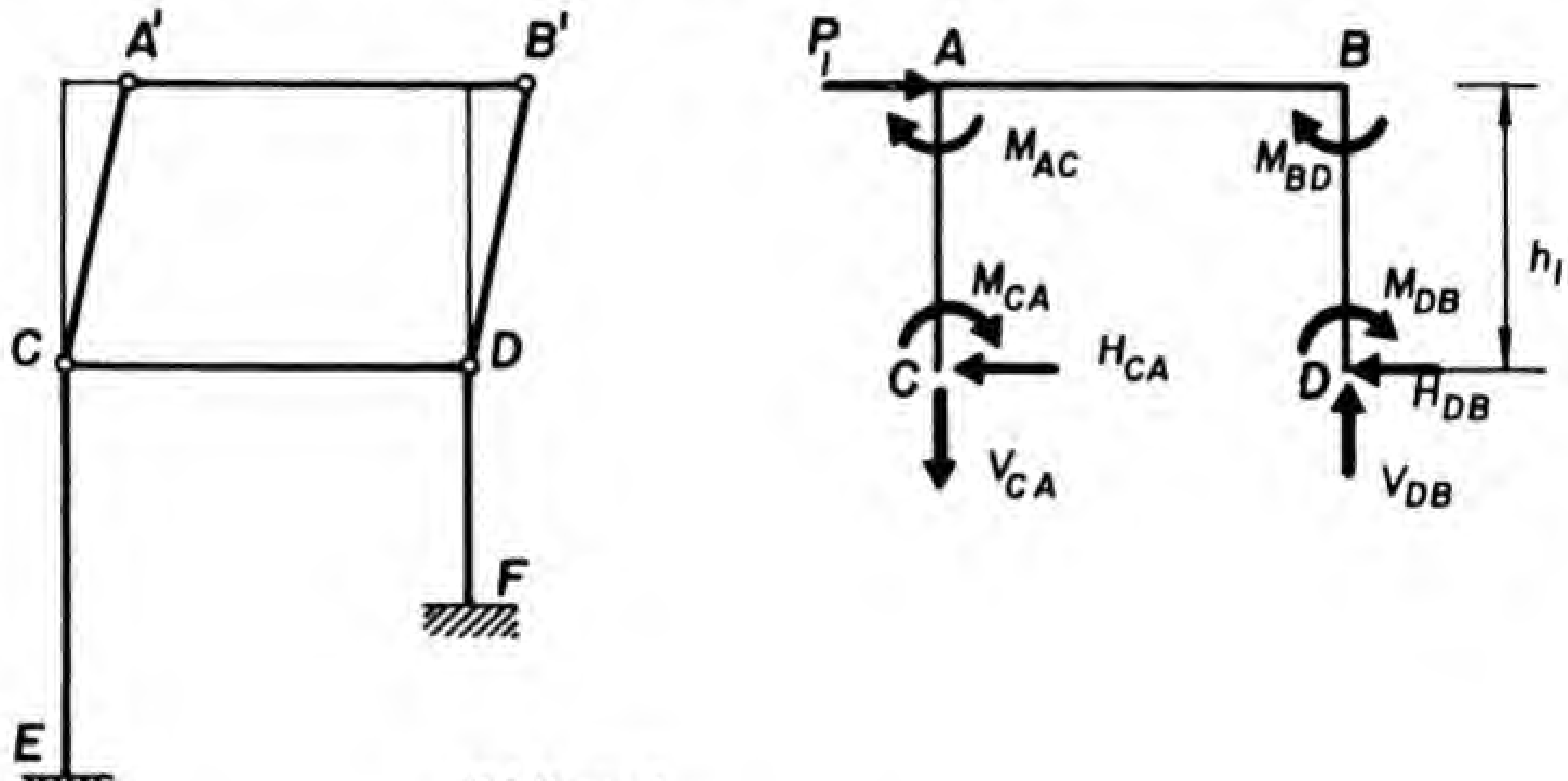
3.5 SWAY EQUATIONS

In simple frames the sway equation was obtained by considering the horizontal static equilibrium of the frame. In more complicated frames, such as two or more storey frames or gabled frames, the equilibrium of the appropriate parts of the frame must be examined. To achieve this the number of independent sway modes must be identified. This may be done by replacing the rigid joints by hinged joints; the number of independent sway modes is then equal to the number of independent kinematic mechanisms. Since an independent mechanism corresponds to the action of different loading systems, each mechanism then corresponds to an independent equation of equilibrium. The sway equations are therefore obtained by considering the equilibrium of the parts of the frame corresponding to the independent mechanism in turn.

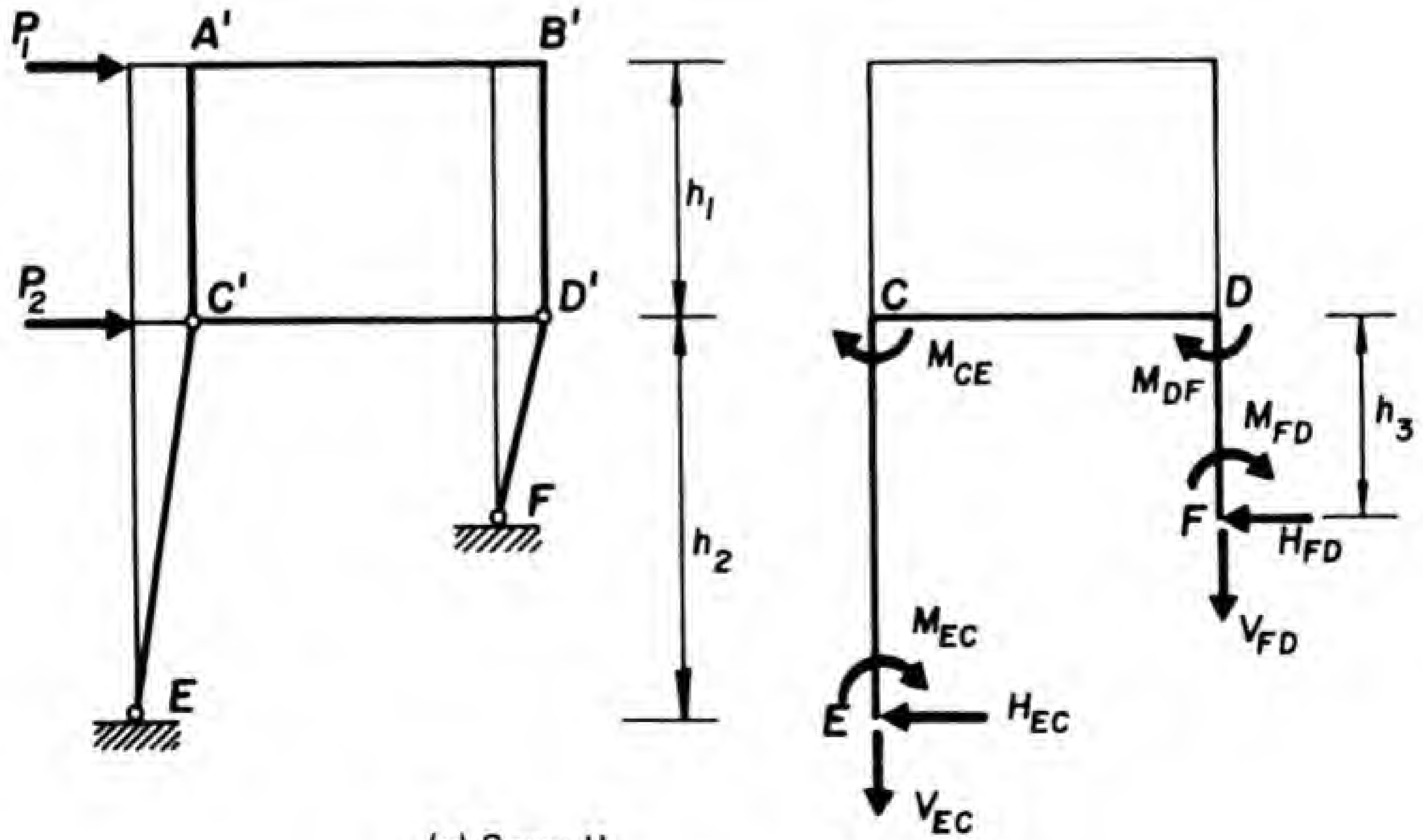
A two-storey frame as shown in Fig. 3.7 is used to illustrate the derivation of



(a) Actual structure



(b) Sway I



(c) Sway II

Figure 3.7

sway equations. The frame has a total of six unknown displacements, namely $\theta_A, \theta_B, \theta_C, \theta_D, \Delta_B$ and Δ_D. The four joints furnish four equations of equilibrium and the additional two equations of equilibrium may be obtained from independent sway modes.

The frame has two independent modes of sway identified by the displacements Δ_1 and Δ_2 as shown in Fig. 3.7(b) and (c). The two independent sway equations are obtained by considering the equilibrium of the portion of the frame corresponding to each mode of sway.

Sway I

In column AC, from $\Sigma M_A = 0$

$$H_{CA} = \frac{M_{AC} + M_{CA}}{h_1}$$

In column BD, from $\Sigma M_B = 0$

$$H_{DB} = \frac{M_{BD} + M_{DB}}{h_1}$$

From $\Sigma H = 0$ of frame ABCD,

$$H_{CA} + H_{DB} = P_1$$

or

$$\frac{1}{h_1}(M_{AC} + M_{CA} + M_{BD} + M_{DB}) = P_1 \qquad [3.14]$$

Sway II

In a similar manner, the relationship between the lateral forces on the whole frame and the end moments is

$$\frac{1}{h_2}(M_{CE} + M_{EC}) + \frac{1}{h_3}(M_{DF} + M_{FD}) = P_1 + P_2 \qquad [3.15]$$

For the case of a gabled frame shown in Fig. 3.8(a), there are two independent mechanisms corresponding to two sway modes as indicated in Fig. 3.8(b) and (c).

$$\psi_{DC} = \psi_{CD} = +\psi_1$$

$$\psi_{AB} = \psi_{BA} = +\frac{2a}{b}\psi_1$$

Since beams BC and CD are equal in length,

$$\psi_{BC} = \psi_{CB} = -\psi_{CD} = -\psi_{DC} = -\psi_1$$

Similarly for Sway II

$$\psi_{BC} = \psi_{CB} = -\psi_{CD} = -\psi_{DC} = +\psi_2$$

$$\psi_{DE} = \psi_{ED} = +\frac{2a}{b}\psi_2$$

Some of the slope deflection equations are:

$$M_{AB} = -2EK_{AB}\left(\theta_B - 3\frac{2a}{b}\psi_1\right)$$

$$M_{BC} = -2EK_{BC}\,[2\theta_B + \theta_D - 3(-\psi_1 + \psi_2)]$$

$$M_{CD} = -2EK_{CD}\,[(2\theta_C + \theta_D - 3(\psi_1 - \psi_2)]$$

$$M_{DE} = -2EK_{DE}\left(2\theta_D - 3\frac{2a}{b}\psi_2\right)$$

The sway equations are determined from the free-body diagrams shown in Fig. 3.8(d) and (e).

The equilibrium equations corresponding to Sway I are:
From $\Sigma M_B = 0$ for member AB,

$$H_A = -\frac{M_{AB} + M_{BA}}{b} \qquad [3.16]$$

From $\Sigma M_C = 0$ for member CD,

$$H_D = -\frac{1}{a}\left(M_{CD} + M_{DC} - \frac{V_D L}{2}\right) \qquad [3.17]$$

From $\Sigma M_B = 0$ for member ABCD

$$V_D = -\frac{1}{L}\left(\frac{VL}{2} + M_{DC} - M_{BA}\right) \qquad [3.18]$$

Substituting [3.17] and using $\Sigma H = P + H_A + H_D = 0$, the sway equation corresponding to Sway I is

$$H - \frac{VL}{4a} = \frac{M_{AB} + M_{BA}}{b} + \frac{M_{BA} + M_{DC}}{2a} + \frac{M_{CD}}{a} \qquad [3.19]$$

In a similar manner for Sway II,

$$\frac{VL}{4a} = \frac{M_{BC} + M_{DE}}{2a} + \frac{M_{DE} + M_{ED}}{b} + \frac{M_{CB}}{a} \qquad [3.20]$$

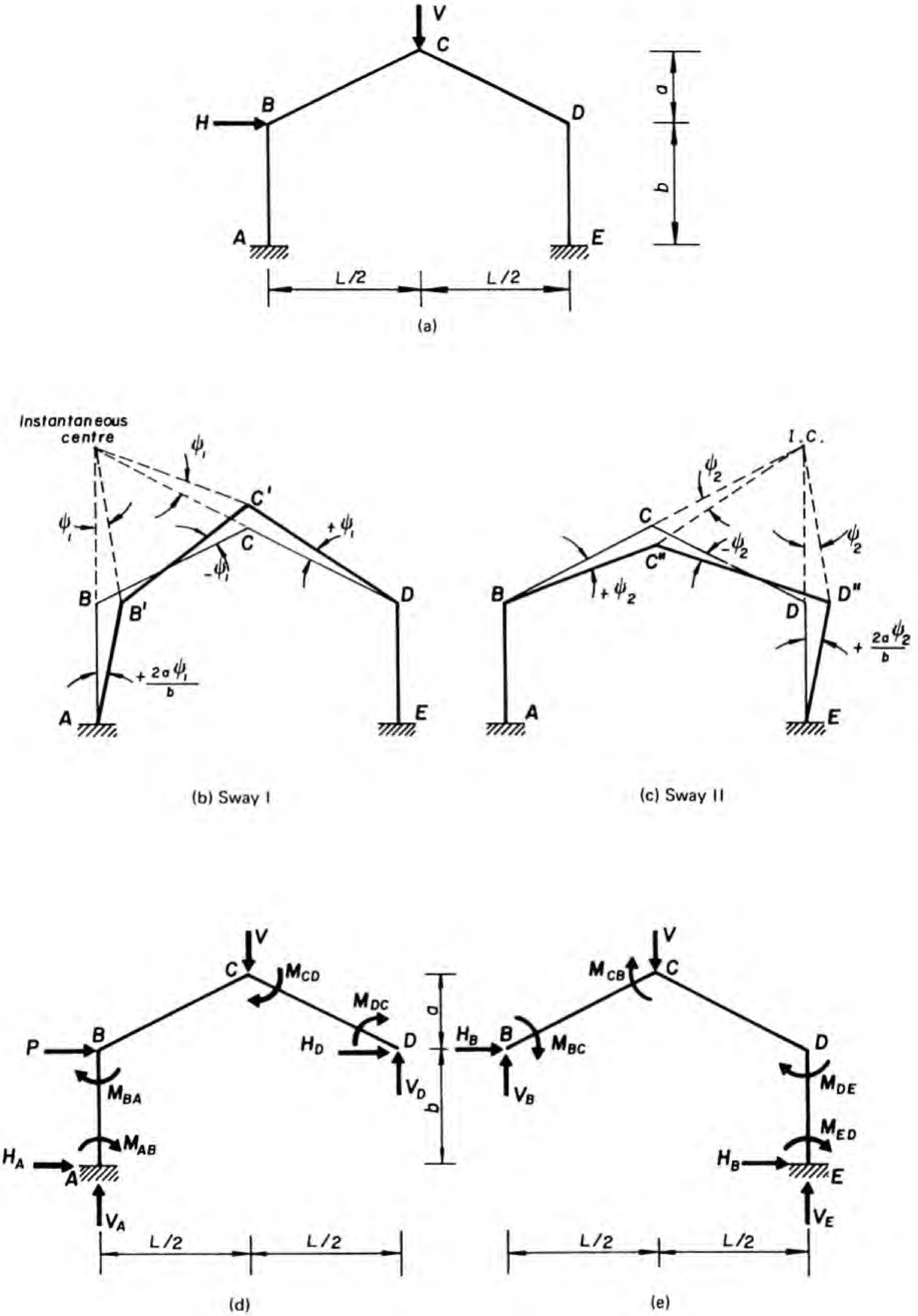
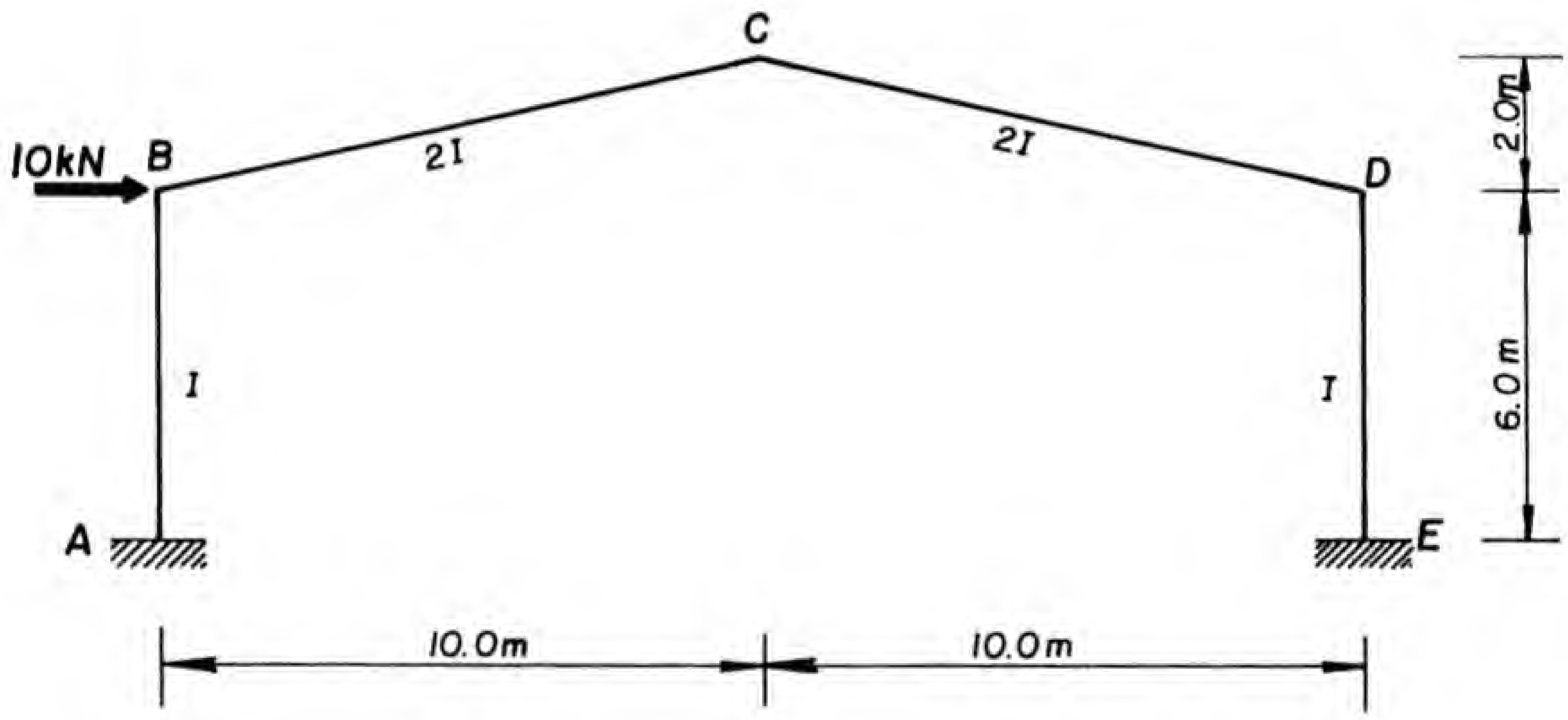


Figure 3.8

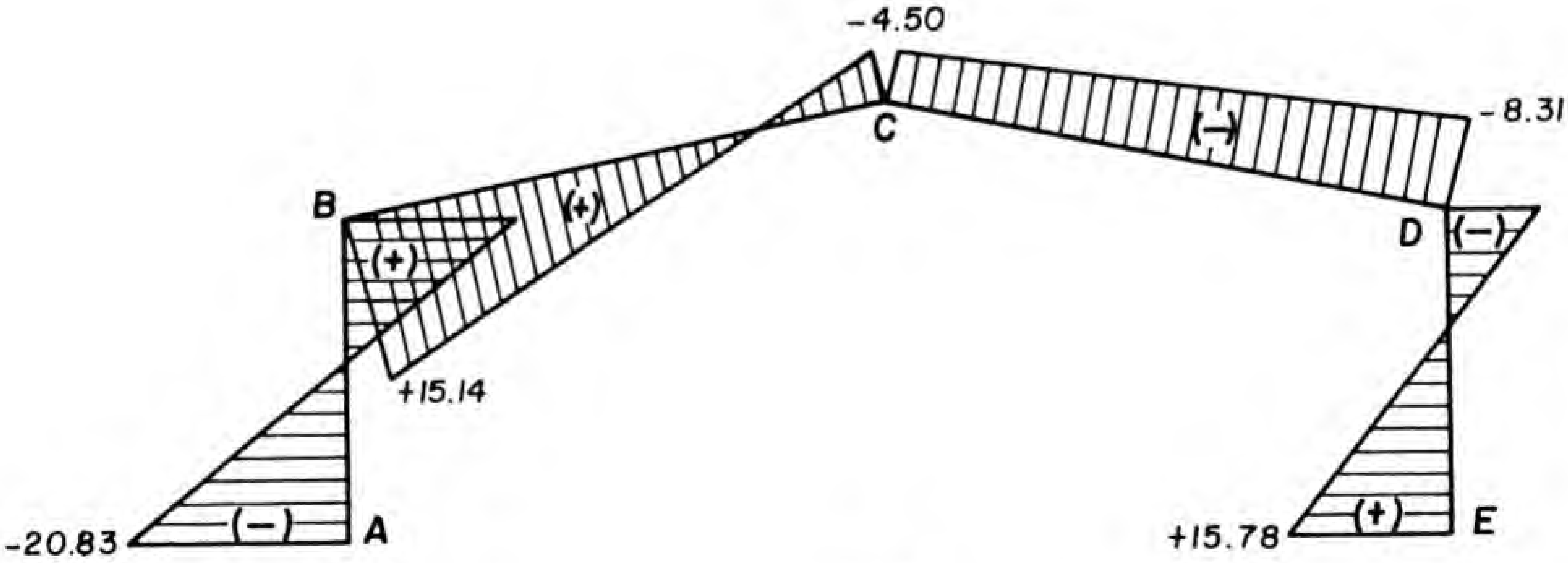
EXAMPLE 3.4 *Determine the joint moments of the gable frame of Fig. 3.9 using slope deflection equations.*

Relative Stiffness

Member	AB	BC	CD	DE
Moment of Inertia	I	$2I$	$2I$	I
Length (m)	6.0	10.2	10.2	6.0
I/L	$I/6$	$I/5.1$	$I/5.1$	$I/6$
Relative K	1.70	2.0	2.0	1.70



(a)



(b)

Figure 3.9

Member Rotations

Sway I:

$$\psi_{AB} = \psi_{BA} = +\frac{2 \times 2}{6}\phi_1 = +0.667\phi_1$$

$$\psi_{BC} = \psi_{CB} = -\phi_1$$

$$\psi_{CD} = \psi_{DC} = \phi_1$$

Sway II:

$$\psi_{BC} = \psi_{CB} = \phi_2$$

$$\psi_{CD} = \psi_{DC} = -\phi_2$$

$$\psi_{DE} = \psi_{ED} = \frac{2 \times 2}{6}\phi_2 = 0.667\phi_2$$

Slope Deflection Equations

Since supports A and E are fixed, $\theta_A = \theta_E = 0$

$$M_{AB} = -1.7(\theta_B - 3 \times 0.667\phi_1)$$

$$M_{BA} = -1.7(2\theta_B - 3 \times 0.667\phi_1)$$

$$M_{BC} = -2.0[2\theta_B + \theta_C - 3(-\phi_1 + \phi_2)]$$

$$M_{CB} = -2.0[2\theta_C + \theta_B - 3(-\phi_1 + \phi_2)]$$

$$M_{CD} = -2.0[2\theta_C + \theta_D - 3(\phi_1 - \phi_2)]$$

$$M_{DC} = -2.0[2\theta_D + \theta_C - 3(\phi_1 - \phi_2)]$$

$$M_{DE} = -1.7(2\theta_D - 3 \times 0.667\phi_2)$$

$$M_{ED} = -1.7(\theta_D - 3 \times 0.667\phi_2)$$

Joint Equilibrium Conditions

(i) Joint B: $M_{BA} + M_{BC} = 0$

(ii) Joint C: $M_{CB} + M_{CD} = 0$

(iii) Joint D: $M_{DC} + M_{DE} = 0$

Shear Conditions

Using [3.19] and [3.20]:

$$\text{(iv)}\quad 10 = \frac{M_{AB} + M_{BA}}{6} + \frac{M_{BA} + M_{DC}}{2 \times 2} + \frac{M_{CD}}{2}$$

$$\text{(v)}\quad 0 = \frac{M_{BC} + M_{DE}}{2 \times 2} + \frac{M_{DE} + M_{ED}}{6} + \frac{M_{CB}}{2}$$

Substituting the moment expressions into the equilibrium equations gives the following simultaneous equations:

$$\begin{aligned}
&-7.4\theta_B - 2.0\theta_C - 2.6\phi_1 + 6\phi_2 = 0\\
&-2.0\theta_B - 8.0\theta_C - 2.0\theta_D = 0\\
&\qquad - 2.0\theta_C - 7.4\theta_D + 6\phi_1 - 2.6\phi_2 = 0\\
&10.2\theta_B + 15\theta_C + 12\theta_D - 38.9\phi_1 + 27\phi_2 = -60\\
&12\theta_B + 15\theta_C + 10.2\theta_D + 27\phi_1 - 38.9\phi_2 = 0
\end{aligned} \qquad [3.21]$$

Solving the five simultaneous equations,

$$\theta_B = 3.350$$

$$\theta_C = -1.948$$

$$\theta_D = 4.443$$

$$\phi_1 = 7.804$$

$$\phi_2 = 6.864$$

End Moments

Substituting the computed values of rotation into [3.21] gives

$$M_{AB} = -1.7(3.350 - 3 \times 0.667 \times 7.804) = +20.84 \text{ kN m}$$

$$M_{BA} = -1.7(2 \times 3.350 - 3 \times 0.667 \times 7.804) = +15.14 \text{ kN m}$$

$$M_{BC} = -2.0[2 \times 3.350 - 1.960 - 3(-7.804 + 6.864)] = -15.14 \text{ kN m}$$

$$M_{CB} = -2.0[-2 \times 1.948 + 3.355 - 3(-7.804 + 6.864)] = -4.55 \text{ kN m}$$

$$M_{CD} = -2.0[-2 \times 1.948 + 4.465 - 3(7.804 - 6.864)] = -4.55 \text{ kN m}$$

$$M_{DC} = -2.0[2 \times 4.443 - 1.960 - 3(7.804 - 6.864)] = -8.23 \text{ kN m}$$

$$M_{DE} = -1.7(2 \times 4.443 - 3 \times 0.667 \times 6.864) = +8.23 \text{ kN m}$$

$$M_{ED} = -1.7(4.443 - 3 \times 0.667 \times 6.864) = +15.78 \text{ kN m}$$

The final bending moment diagram is shown in Fig. 3.9(b).

3.6 PROBLEMS

3.1 Find the joint moments of the box structure shown in Fig. P3.1.

(*Ans*: $M_A = M_D = -0.24$ kN m

$M_B = M_C = -0.88$ kN m)

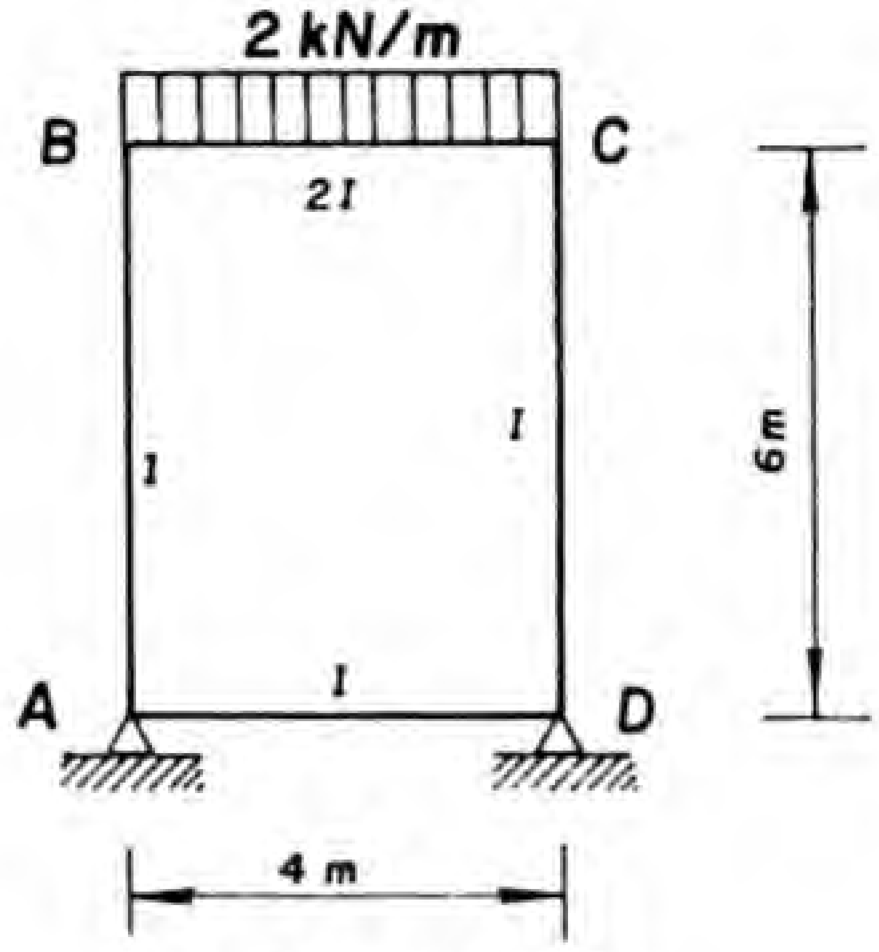


Figure P3.1

3.2 Determine the joint moments of the frame shown in Fig. P3.2.

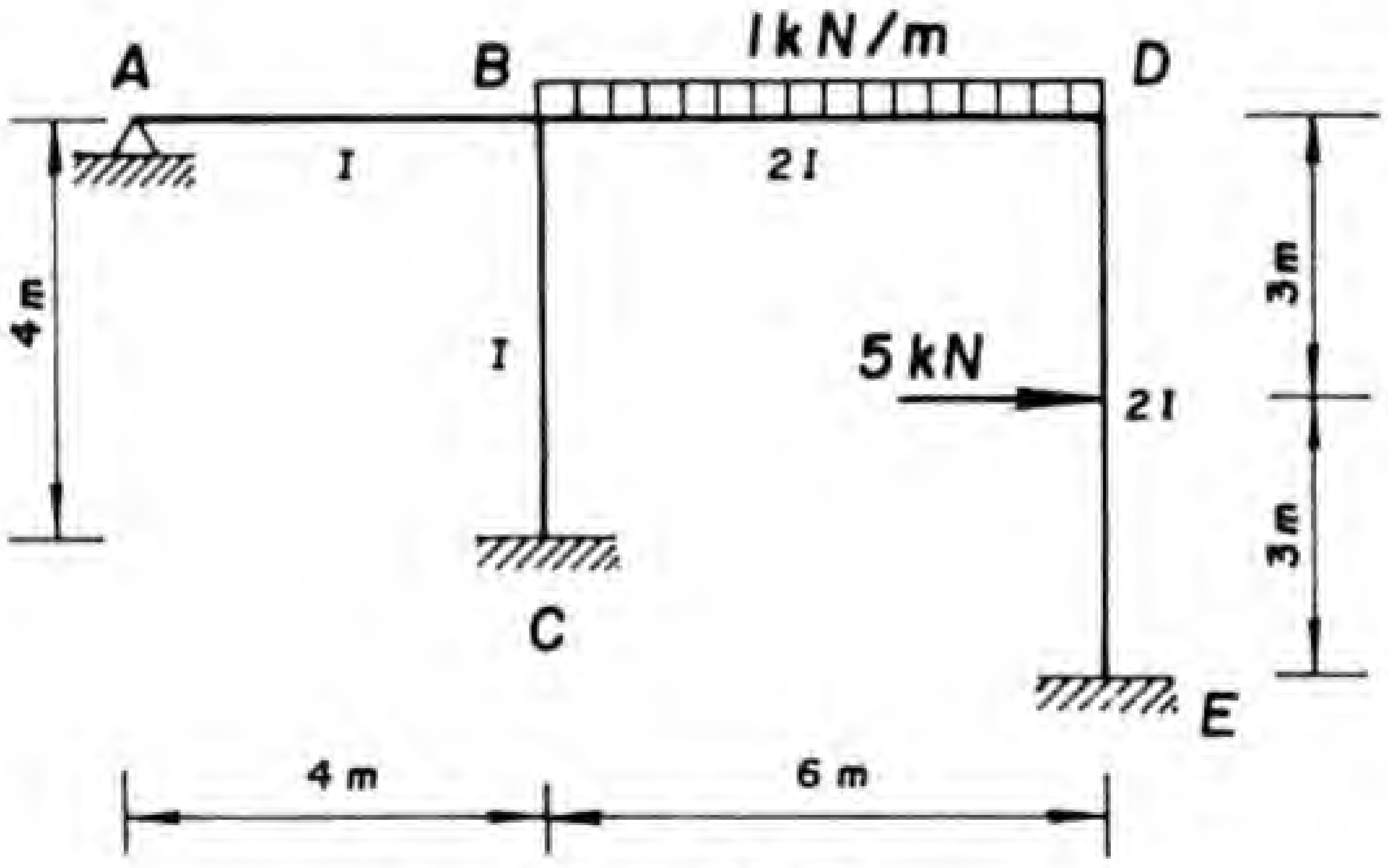


Figure P3.2

3.3 Draw the bending moment diagram of the frame shown in Fig. P3.3.

(*Ans*: $M_A = -6.60$ kN m

$M_B = +6.08$ kN m

$M_C = -10.09$ kN m and -0.09 kN m

$M_D = +7.19$ kN m)

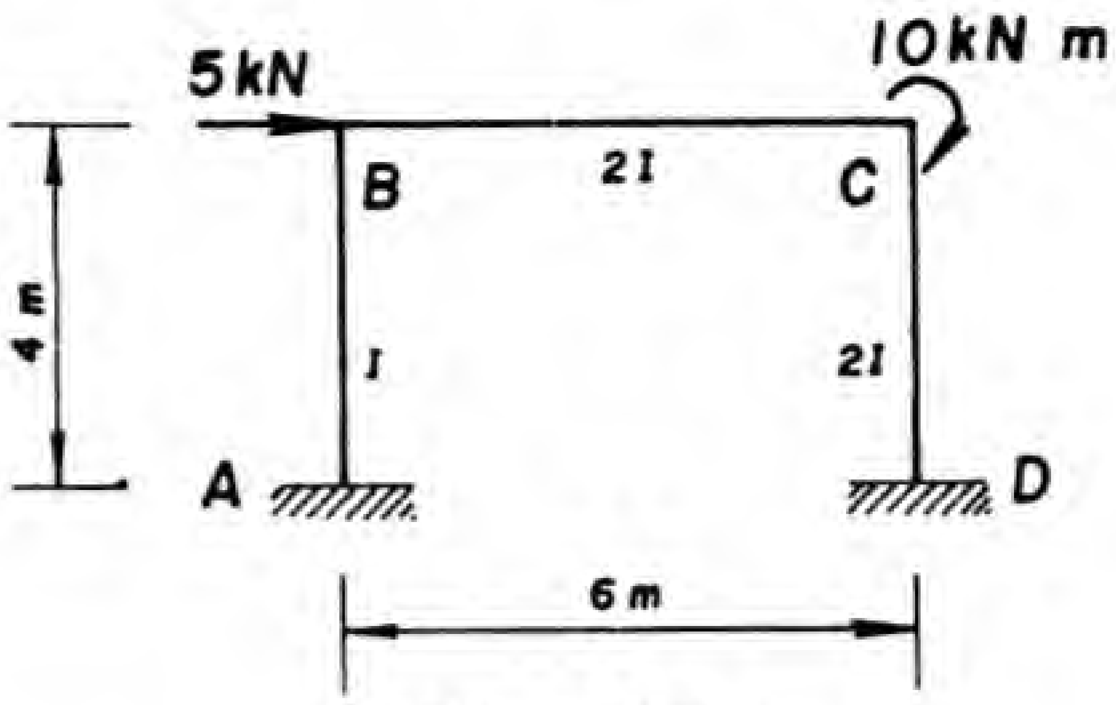


Figure P3.3

3.4 Find the joint moments of the frame shown in Fig. P3.4.

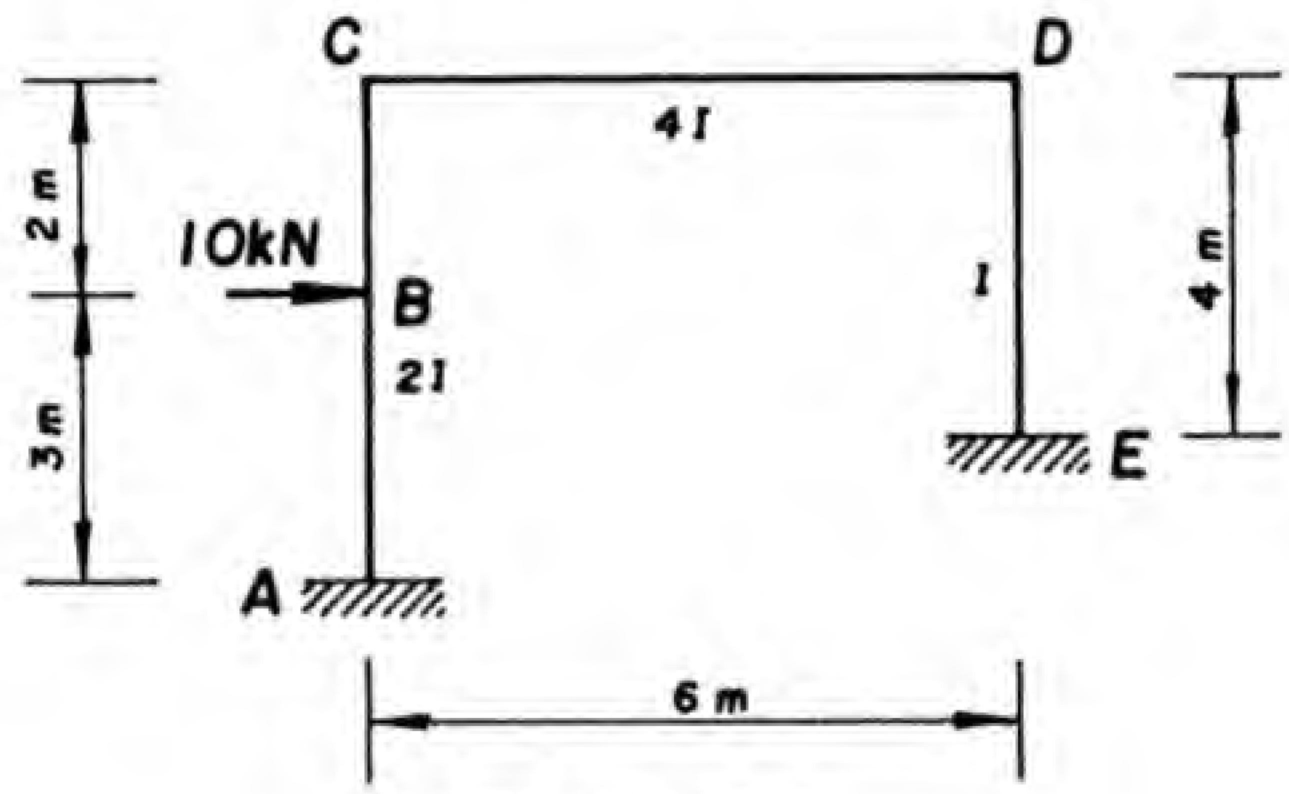


Figure P3.4

3.5 Find the joint moments of the gable frame shown in Fig. P3.5.

(*Ans*: M_B = +4.66 kN m

M_B = –6.15 kN m and –8.15 kN m

M_C = +4.45 kN m)

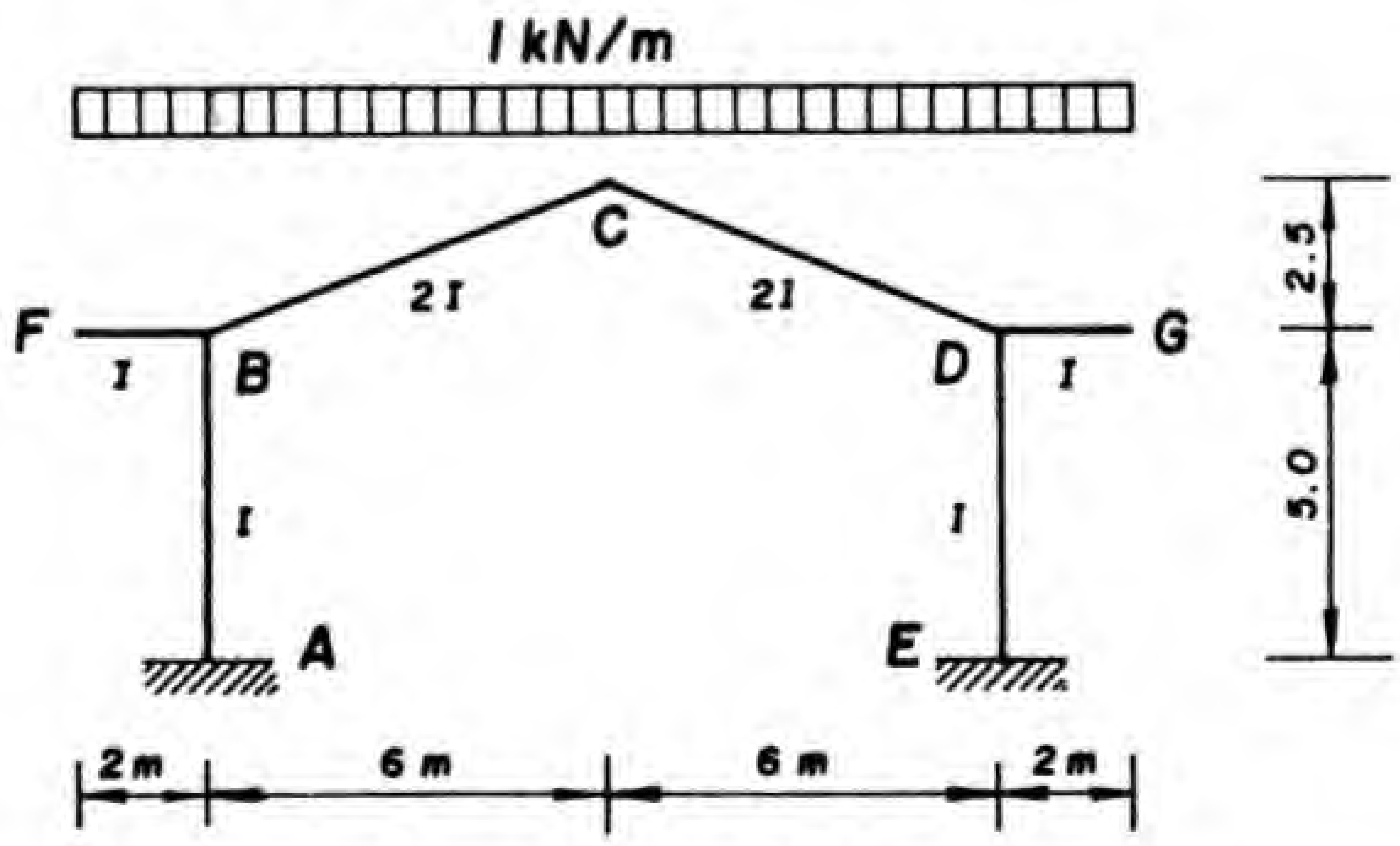


Figure P3.5

3.6 Draw the bending moment diagram of the gable frame shown in Fig. P3.6.

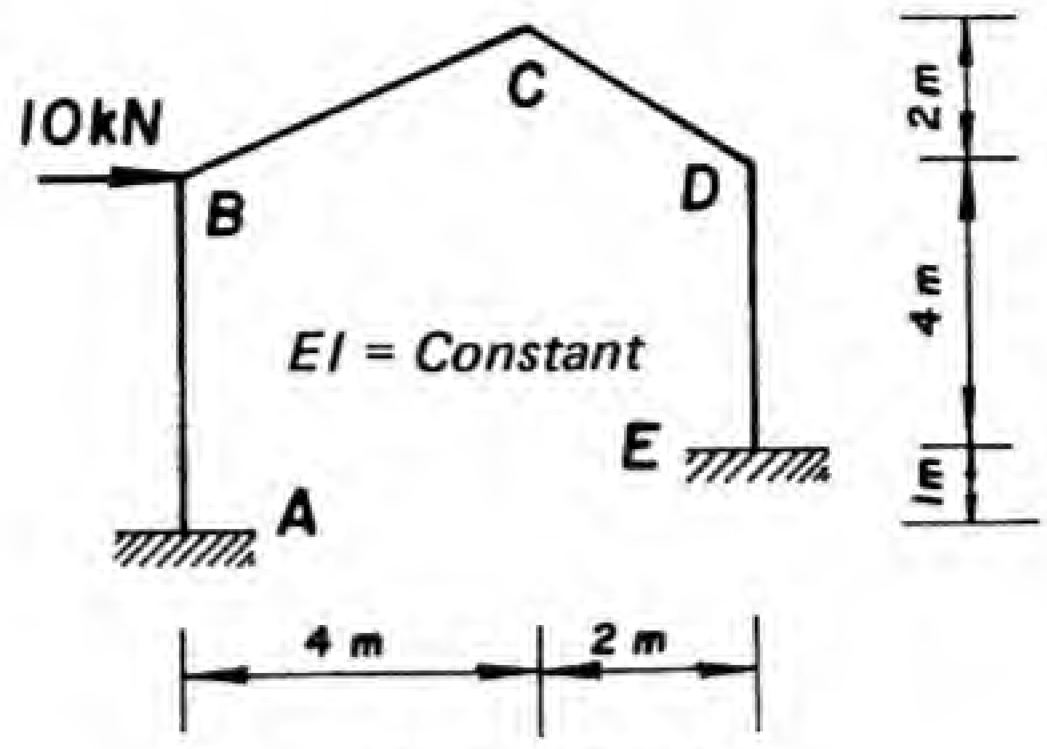


Figure P3.6

4. The Cross Method of Moment Distribution

4.1 INTRODUCTION

The moment distribution method was developed by Hardy Cross and was formally presented in 1930. The method became dominant for continuous beams and rigid frame analysis for the next thirty years. The method remains very important for manual solutions of frames. The method of solution places emphasis on physical behaviour, which is performed in tabular form, and is quick and easily remembered. Because of the nature of the concept and procedure, which readily lends itself to the development of design judgement, the method has transformed engineering thinking and design office practices.

The moment distribution method starts from the same basic assumption made in the slope deflection method. In the analysis of continuous beams and frames all joints are assumed fixed and the moments are then *corrected* by means of the slope deflection equations. While in the slope deflection method the correction is effected by solving a set of linear simultaneous equations, in the moment distribution method the *correction* is dealt with by successive approximations through the use of arithmetic only where the results can be obtained to any desired accuracy.

4.2 ITERATIVE SOLUTION OF SLOPE DEFLECTION EQUATIONS

The slope deflection equation involves the formulation of equations of joint equilibrium in terms of rotations, displacements, stiffness and length of members. These equations of equilibrium form a set of linear simultaneous equations which may be solved by methods such as elimination methods, iteration methods, or relaxation techniques.

One of the iteration methods for obtaining the solution of a set of simultaneous equations is by the Gauss–Seidel method. To illustrate the method, consider

the problem of Example 3.1. The governing equations of equilibrium are

$$26\theta_B + 8\theta_C = -64$$

$$8\theta_B + 16\theta_C = -144$$

The above equations may be written as

$$\theta_B = -2.462 - 0.308\theta_C \qquad [a]$$

$$\theta_C = -9.0 - 0.5\theta_B \qquad [b]$$

One means of obtaining an approximate solution to the equations is to make a guess. Substituting the assumed values in the right-hand sides of [a] and [b] furnishes a new set of values on the left. If the first assumed values do not work, the new set of values are selected as a better guess and the process is repeated until a consistent set of values are obtained that satisfy the equations. To furnish a more rapid convergence of the iterative process, the Gauss–Seidel method presupposes the most recent cycle of values at each stage of calculations. This method is explained below by solving the above equations.

Cycle 1 Assume $\theta_C = 0$ in [a] and solve; $\theta_B = -2.462$. Using $\theta_B = -2.462$ in [b], solve; $\theta_C = -7.769$.

Cycle 2 Using the new value of $\theta_C = -7.768$ in [a] obtain a new value for θ_B (−0.069). Use $\theta_B = -0.069$ in [b] to solve; $\theta_C = -8.965$.

This procedure continues, as shown in Table 4.1, for a sufficient number of cycles until two successive cycles agree within acceptable error-limits. The final results are

$$\theta_B = +0.366$$

$$\theta_C = -9.183$$

which agree closely with those obtained in Example 3.1

Table 4.1

Cycle	θ_B	θ_C
Initial	0	0
1	−2.462	−7.769
2	−0.069	−8.965
3	+0.299	−9.150
4	+0.356	−9.178
5	+0.365	−9.182
6	+0.366	−9.183
7	+0.366	−9.183

4.3 INTERPRETATION OF THE ITERATIVE SOLUTION

Each iterative cycle shown in Section 4.2 has a physical interpretation on the beam of Example 3.1. It is recalled that the slope deflection equations are

$$\begin{aligned} M_{AB} &= 160 - 5\theta_B \\ M_{BA} &= -160 - 10\theta_B \\ M_{BC} &= 96 - 16\theta_B - \theta_C \\ M_{CB} &= -144 - 8\theta_B - 16\theta_C \end{aligned} \qquad [c]$$

and the equilibrium equations are

$$M_{BA} + M_{BC} = 0 \qquad [d]$$

$$M_{CB} = 0 \qquad [e]$$

The values of θ_B and θ_C for each step of the iteration scheme as evaluated in Table 4.1 are given in Table 4.2. Also shown are the moments of the slope deflection equations corresponding to the values of θ_B and θ_C evaluated in Table 4.1

Table 4.2

Cycle	θ_B	θ_C	M_{AB}	M_{BA}	M_{BC}	M_{CB}
Initial	0	0	160	−160	96	−144
1	−2.462	0 −7.769	172.31	−135.38	135.39 197.54	−124.30 0.0
2	−0.069	−7.769 −8.965	160.35	−159.31	159.26 168.82	−19.14 −0.01
3	+0.299	−8.965 −9.150	158.51	−162.09	162.94 164.42	−2.95 +0.01
4	+0.356	−9.150 −9.178	158.22	−163.56	163.50 163.73	−0.45 0.00
5	+0.365	−9.178 −9.183	158.18	−163.65	163.58 163.62	−0.07 −0.01

A close review of the joint rotations and the corresponding support moments in Table 4.2 reveals the following:

(a) At the initial stage none of the equilibrium equations, [d] or [e], is satisfied.

(b) In the first phase of Cycle 1, joint B is permitted to rotate while joint C remains clamped. Consequently, [d] is satisfied while [e] is not.
(c) In the second phase of Cycle 1, joint B is clamped in its rotated position while joint C is allowed to rotate until M_{CB} becomes zero, thus satisfying [e]. However [d] is not satisfied.
(d) In Cycle 2, as in Cycle 1, the same pattern of permitting rotation at one support while clamping at the other is repeated.
(e) The process of permitting joint rotations at one support while clamping the other is repeated as many times until the equilibrium equations are satisfied. At this stage the correct support moments are obtained.

This process of balancing operation by alternately permitting and preventing joint rotations at each joint until the equilibrium condition is attained is the physical process which corresponds to the iterative solution. This physical process is known as the *moment distribution method*, and it is developed hereafter.

4.4 FUNDAMENTAL FACTORS USED IN MOMENT DISTRIBUTION

Prior to developing the method of moment distribution, certain definitions and fundamental relationships need to be considered.

Rotational Stiffness

Consider a beam with constant moment of inertia, I, and no interior load (Fig. 4.1). If end B is fixed and there is no relative displacement of the ends, the slope deflection equation is written as:

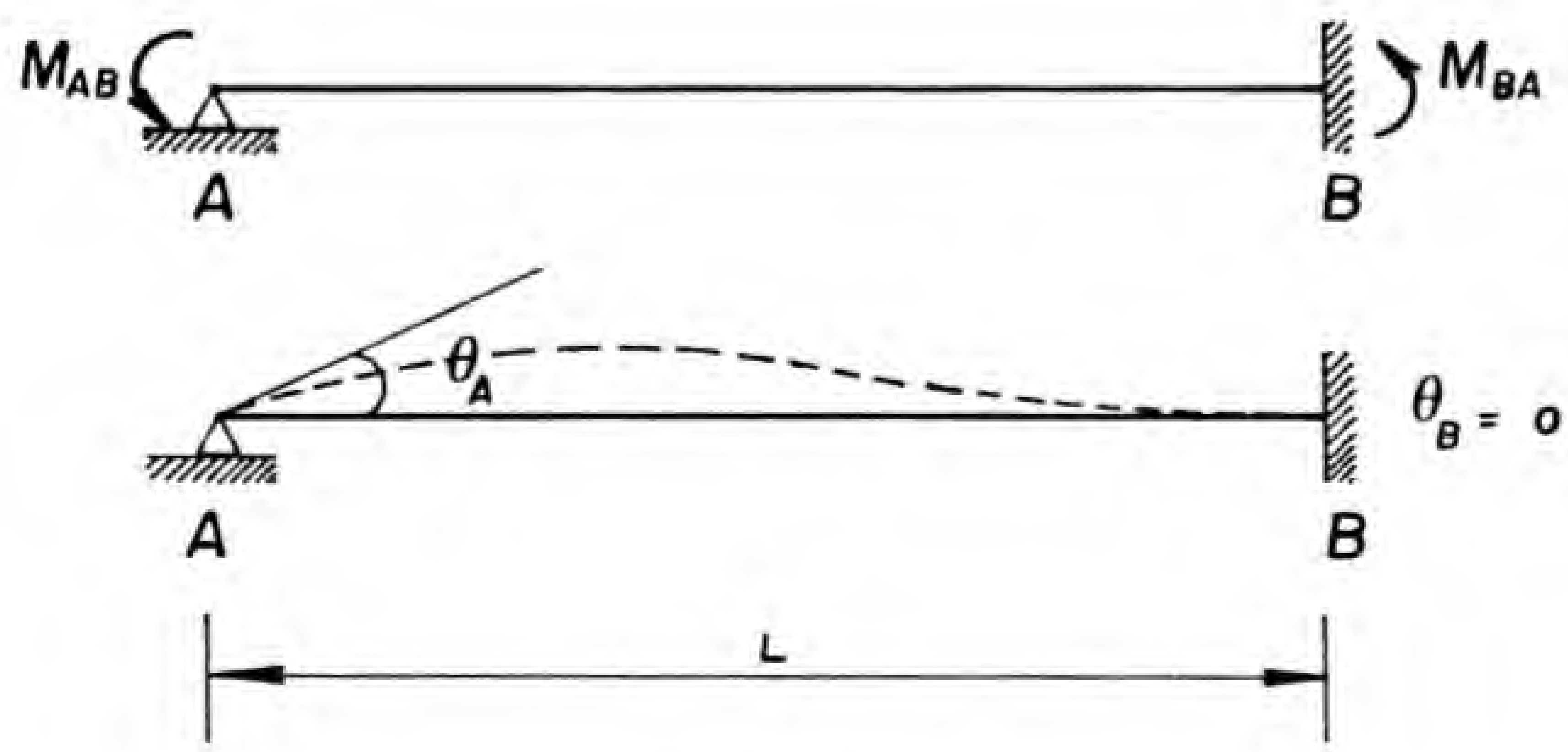


Figure 4.1

$$M_{AB} = 2EK(2\theta_A)$$

or

$$\frac{M_{AB}}{\theta_A} = 4EK \qquad [4.1]$$

Since the value of M_{AB}/θ_A is a measure of the resistance of the beam to rotation, it is defined as the *stiffness* of the member.

When a beam has its far end hinged instead of fixed, less moment is required to rotate the end through a given angle. Figure 4.2 is simply supported at the far end and a moment M_{AB} is applied at the near end. By the slope deflection method,

$$M_{AB} = -2EK(2\theta_A - \theta_B)$$

$$M_{BA} = -2EK(2\theta_B + \theta_A) = 0$$

or

$$\theta_B = -\theta_A/2$$

Thus

$$M_{AB} = 3EK\theta_A \qquad [4.2]$$

It is noted that the stiffness of the beam at the near end when the far end is hinged is *three-fourths* of the stiffness when the far end is fixed. The significance of this ratio is that it permits the establishment of distribution factors consistent with true end hinged taken as $3EI/L$; the far end can be released in the first balancing operation and then left *free* during subsequent calculations, or, no moment is carried over to the hinged end. The adjusted relative stiffness makes this unnecessary and the technique substantially reduces the calculation.

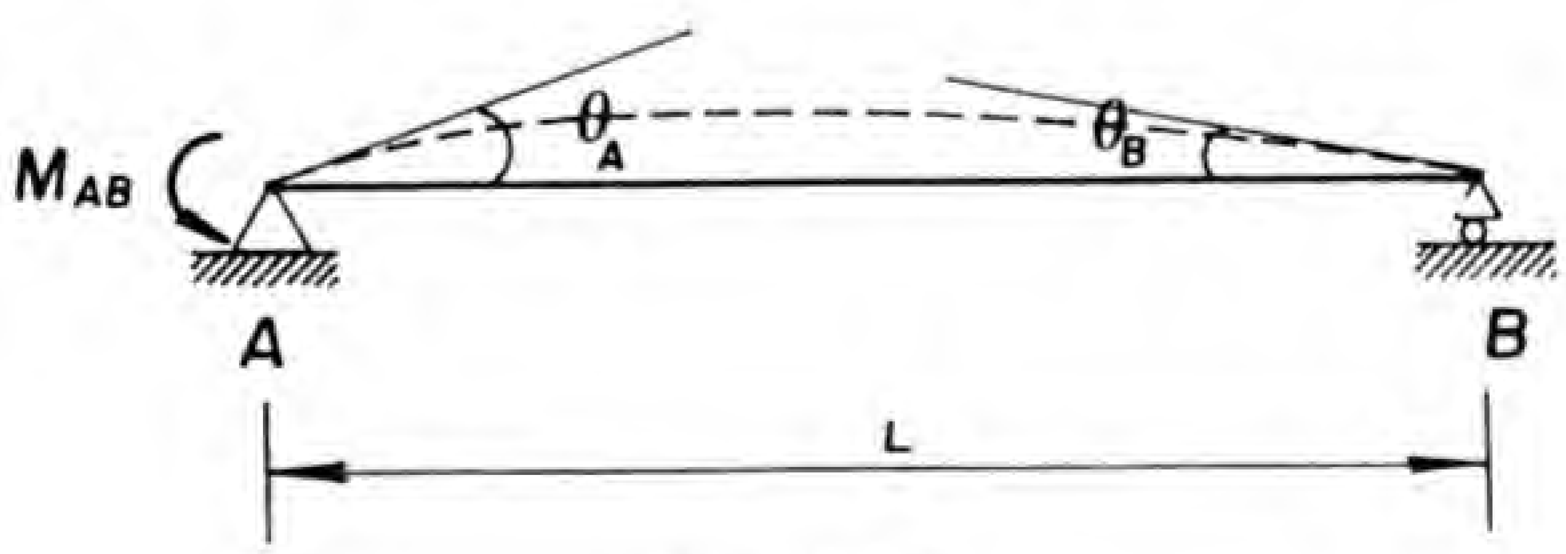


Figure 4.2

Carry-Over Factor

The slope deflection equation gives the corresponding moment at the far end, Fig. 4.1 (support B), as

$$M_{BA} = 2EK\theta_A$$

The ratio of the induced moment M_{BA} to the applied moment M_{AB}

$$\frac{M_{BA}}{M_{AB}} = \frac{1}{2} \tag{4.3}$$

is defined as the *carry-over* factor.

Distribution Factor

If there are several members framed into a joint, and if a moment is applied to joint A as shown in Fig. 4.3, then

$$M_{AB} + M_{AC} + M_{AD} - M_o = 0$$

or,

$$M_o = 4E\theta_A(K_{AB} + K_{AC} + K_{AD})$$
$$= 4E\theta_A \Sigma K$$

The moments of the members at joint A are

$$M_{AB} = 4EK_{AB}\theta_A$$
$$M_{AC} = 4EK_{AC}\theta_A$$
$$M_{AD} = 4EK_{AD}\theta_A$$

Substituting $4E\theta_A = M_o/\Sigma K$

$$M_{AB} = \frac{K_{AB}}{\Sigma K}M_o = (DF)_{AB}M_o$$

$$M_{AC} = \frac{K_{AC}}{\Sigma K}M_o = (DF)_{AC}M_o \tag{4.4}$$

$$M_{AD} = \frac{K_{AD}}{\Sigma K}M_o = (DF)_{AD}M_o$$

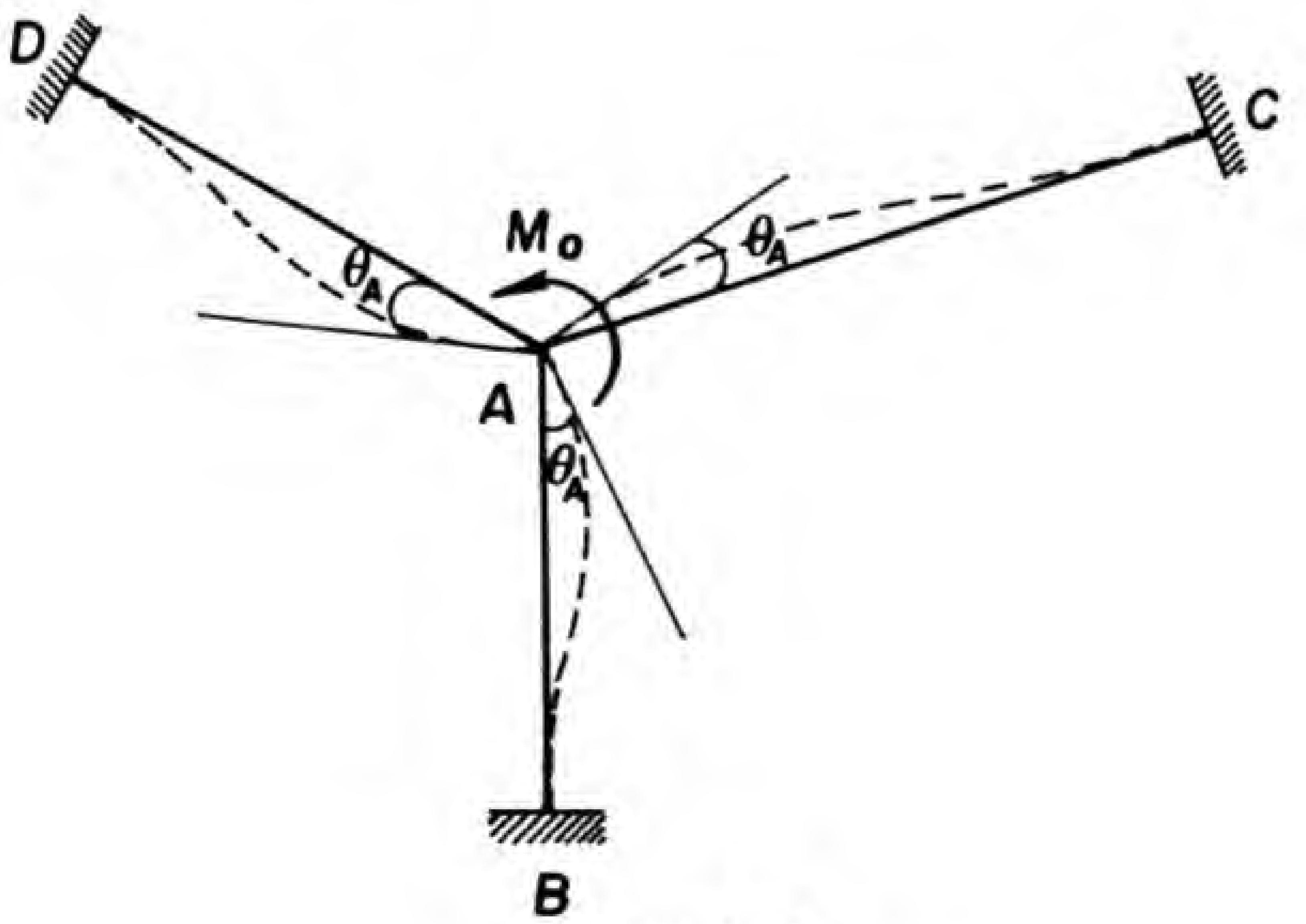


Figure 4.3

Thus the applied moment M_o is distributed to the members in proportion to the relative stiffness K. The term $K/\Sigma K$ is known as the *distribution factor*, which is denoted as DF. It is self-evident that the sum of all distribution factors at a given joint must equal unity.

4.5 MOMENT DISTRIBUTION METHOD FOR BEAM ANALYSIS

The following example is used to illustrate the *how* and *why* of the moment distribution method. The statical moment sign convention is used; that is, positive moment tends to rotate the joint clockwise or the member counter-clockwise.

EXAMPLE 4.1 *Determine the support moment of the continuous beam in Fig. 4.4. The beam has constant moment of inertia I.*

A step-by-step solution is given consisting of (i) locking all joints to prevent rotation, (ii) computing fixed-end moments, (iii) distributing unbalanced moments in proportion to the relative stiffness, (iv) carrying over one-half of the distributed moments to the opposite ends of the members.

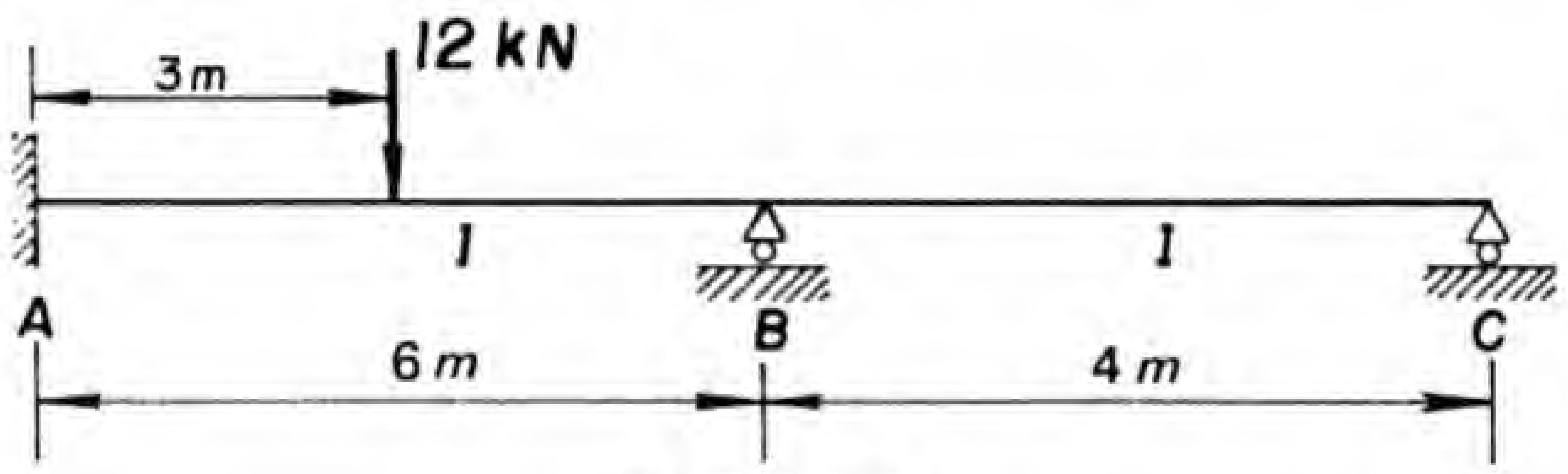


Figure 4.4

Relative Stiffness Values and Distribution Factors

The relative values of $K = I/L$ are:

$$\text{Span AB:} \quad K_{AB} = \frac{1}{6} \times 12 = 2$$

$$\text{Span BC:} \quad K_{BC} = \frac{1}{4} \times 12 = 3$$

The distribution factors are:

$$(DF)_{AB} = \frac{K_{AB}}{\Sigma K} = \frac{2}{2 + \infty} = 0$$

$$(DF)_{BA} = \frac{K_{BA}}{\Sigma K} = \frac{2}{2 + 3} = 0.4$$

$$(DF)_{BC} = \frac{K_{BC}}{\Sigma K} = \frac{3}{2+3} = 0.6$$

$$(DF)_{CB} = \frac{3}{3+0} = 1.0$$

Note that $(DF)_{AB} = 0$ since a fixed joint is considered infinitely stiff and $(DF)_{CB} = 1.0$ since CB is the only member at hinge C. The DF values are entered at the top of the calculation scheme in appropriate places.

Fixed-End Moment

Assume that all joints are *locked* against rotation before the loads are applied on the beam. The fixed-end moments produced by the loads are

$$M_{BA}^{F} = \frac{12 \times 3 \times 3^2}{6^2} = +9.0 \text{ kN m}$$

$$M_{BA}^{F} = -9.0 \text{ kN m}$$

Balancing of Joint Moments and Carry-Over Moments

Distribute the unbalanced moment +9.0 at support B in proportion to the DF values by applying +3.6 on member BA and +5.4 on member BC. Consequently, half the moment of 3.6 kN m is carried to A while half the moment of 5.4 kN m is carried to C. Since support A is fixed, any moment carried over to A remains as carried over. However, joint C is released by *applying* −2.7 kN m to cancel the moment of +2.7; this results in zero moment at C which must be the case for a hinge. As C rotates under the applied moment, B remains fixed and the unbalanced moment at B is −1.35 kN m. The procedure of distributing and carry-over is repeated until all joints are balanced to the desired accuracy. After the final balancing operation the end-moments are algebraically added to get the final end moments shown in Table 4.3.

The support moments are:

$$M_{AB} = 11.11 \text{ kN m}$$

$$M_{BA} = -4.77 \text{ kN m}$$

$$M_{BC} = +4.77 \text{ kN m}$$

$$M_{CB} = 0$$

(see the numbers in italic in Table 4.14)

The balancing computations may also be performed in tabular form as given in Table 4.4.

Table 4.3

Relative K	2		3	
Distribution Factor	0	0.4	0.6	1.0
Fixed-end moment	+9.0	−9.0	0.0	0.0
Distribute		+3.6	+5.4	
Carry-over	+1.8			+2.7
Distribute				−2.7
Carry-over			−1.35	
Distribute		+0.54	+0.81	
Carry-over	+0.27			+0.40
Distribute				−0.40
Carry-over			−0.20	
Distribute		+0.08	+0.12	
Carry-over	+0.04			+0.06
Distribute				−0.06
Carry-over			−0.03	
Distribute		+0.01	+0.02	
Final Moment	+11.11	−4.77	−4.77	0.0

Table 4.4

Joint	A	B		C
Member	AB	BA	BC	CB
K	4	4	6	6
DF	0	0.4	0.6	1.0
Fixed-end moment	+9.00	−9.00		
	+1.80	+3.60	+5.40	+2.70
			−1.35	−2.70
	+0.27	+0.54	+0.81	+0.40
			−0.20	−0.40
	+0.04	+0.08	+0.12	+0.06
			−0.03	−0.06
		+0.01	+0.02	
Total	+11.11	−4.77	+4.77	0.00

EXAMPLE 4.2 *Determine the support moments of the beam given in Example 4.1 using the simplified treatment of a hinge.*

Relative Stiffness Values and Distribution Factors

$$K_{AB} = \frac{12}{6} = 2$$

$$K_{BC} = \frac{3}{4} \times \frac{12}{4} = 2.25$$

$$(DF)_{BA} = \frac{2}{2 + 2.5} = 0.47$$

$$(DF)_{BC} = \frac{2.5}{2 + 2.5} = 0.53$$

The moment distribution is performed in tabular form as shown in Table 4.5.

Table 4.5

Joint	A	B		C
Member	AB	BA	BC	CB
K	2	2	2.5	2.5
DF	0	0.47	0.53	1.0
Fixed-end moment	+9.0 +2.11	+4.23	+4.77	
Total	+11.11	−4.77	+4.77	0.0

For beams with a loaded overhang at the hinged end the stiffness is $3EI/L$ since the stiffness of the cantilever is zero. When there is no load on the cantilever the moment at the hinged end will be zero, but if a load is applied on the cantilever the bending moment will be statically determinate and is independent of loads on adjacent spans. A joint with an overhang, therefore, may be treated as a beam with a hinged end as far as the moment distribution is concerned.

Beams with Relative Displacement of Supports

The moment distribution may be used to analyse continuous beams with relative displacement of supports or rotation of supports. The effects of relative

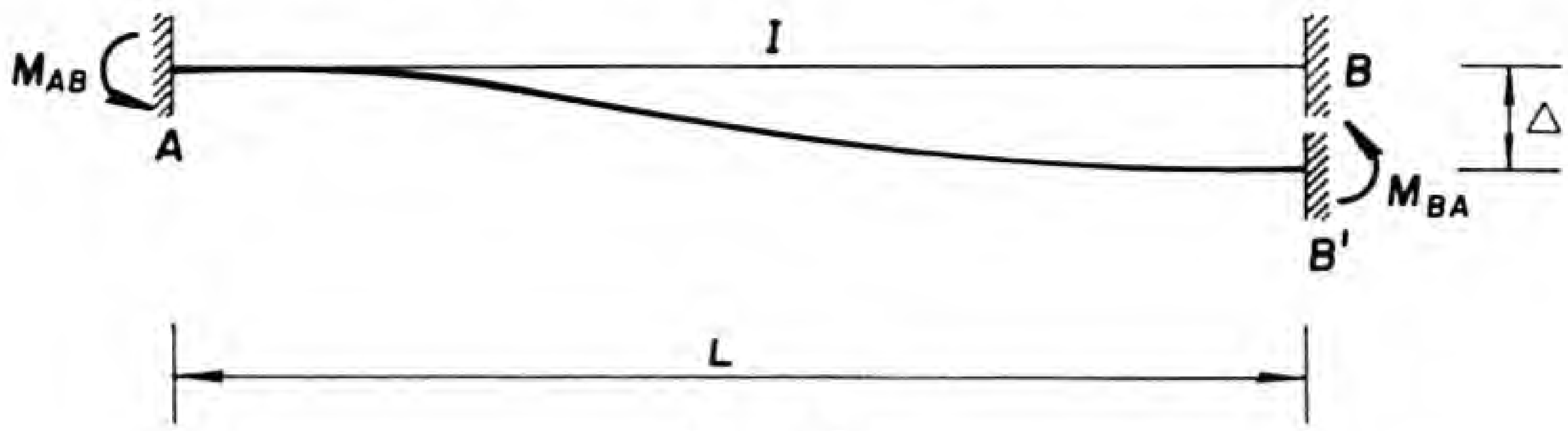


Figure 4.5

displacement of supports can be considered by referring to Fig. 4.5. The beam AB is fixed against rotation at A and B. The end B is deflected downward an amount Δ while both ends are restrained against rotation.

The slope deflection equation is used to determine the value of M_{AB}.

$$M_{AB} = M_{BA} = -2EK(-3\Delta/L)$$

$$= \frac{6EK\Delta}{L}$$

The corresponding *fixed-end* moments are

$$M_{AB}^F = M_{BA}^F = \frac{6EI\Delta}{L^2} \qquad [4.5]$$

Similarly, the fixed-end moments due to rotation θ_A at A are

$$M_{AB}^F = 2M_{BA}^F = \frac{4EI\theta_A}{L} \qquad [4.6]$$

A beam with one end hinged which deflects Δ is shown in Fig. 4.6.

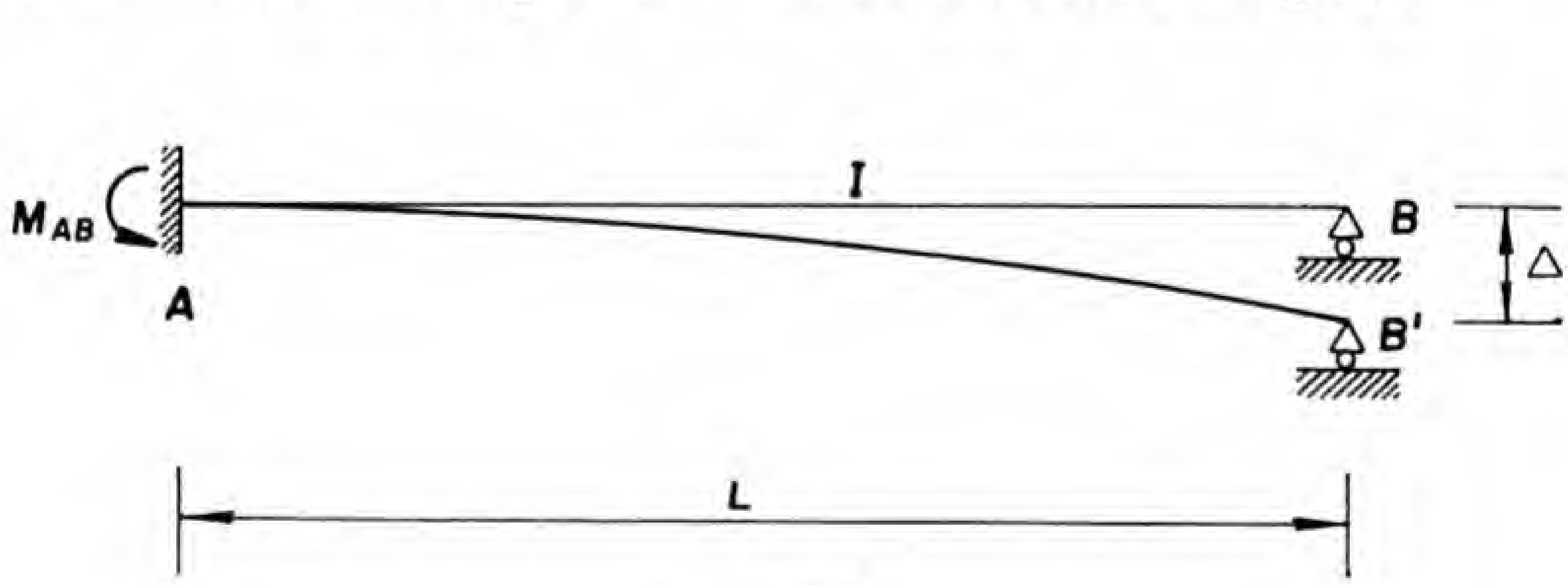


Figure 4.6

The slope deflection equation gives

$$M_{BA} = -2EK(2\theta_B - 3\Delta/L) = 0$$

$$\theta_B = 3\Delta/2L$$

Thus

$$M_{AB} = -2EK\left(\theta_B - \frac{3\Delta}{L}\right)$$

$$= \frac{3EK\Delta}{L^2}$$

The corresponding *fixed-end* moment is

$$M_{AB}^F = \frac{3EI\Delta}{L^2}$$

$$M_{BA} = 0 \tag{4.7}$$

The appropriate fixed-end moment corresponding to the relative displacements will then be introduced into the moment distribution scheme. These moments are treated in exactly the same manner as those due to the applied loadings.

4.6 MOMENT DISTRIBUTION METHOD FOR FRAME ANALYSIS

In applying the moment distribution method to frame problems, there is no inherent limitation which prevents the use of the basic concepts developed so far. The method can be applied to frame problems with some changes in the book keeping scheme. For frames with fewer number of joints, the tabular scheme of calculations may be more convenient to adopt. However, for frames with numerous joints the sketch of the structure itself may be more practical to use. One system of recording the moments is to place the computed value *under* the beam on the *left* and *over* the beam on the *right*. Similarly, the column moments are recorded to the *left* on the *top* of the column and to the *right* on the *bottom*. This arrangement of showing the calculations on the structure itself is shown in Fig. 4.7.

Frame problems are classified in two categories:

(a) Frames without sidesway
(b) Frames with sidesway

4.6.1 Frames Without Sidesway

Frames without sidesway are analysed in the same manner as continuous beams. In the case of frames there are frequently more than two members meeting at a joint so that the joint moments distribute among all members according to the appropriate distribution factors. Consequently, for such frames there is a need of a more systematic arrangement of computations.

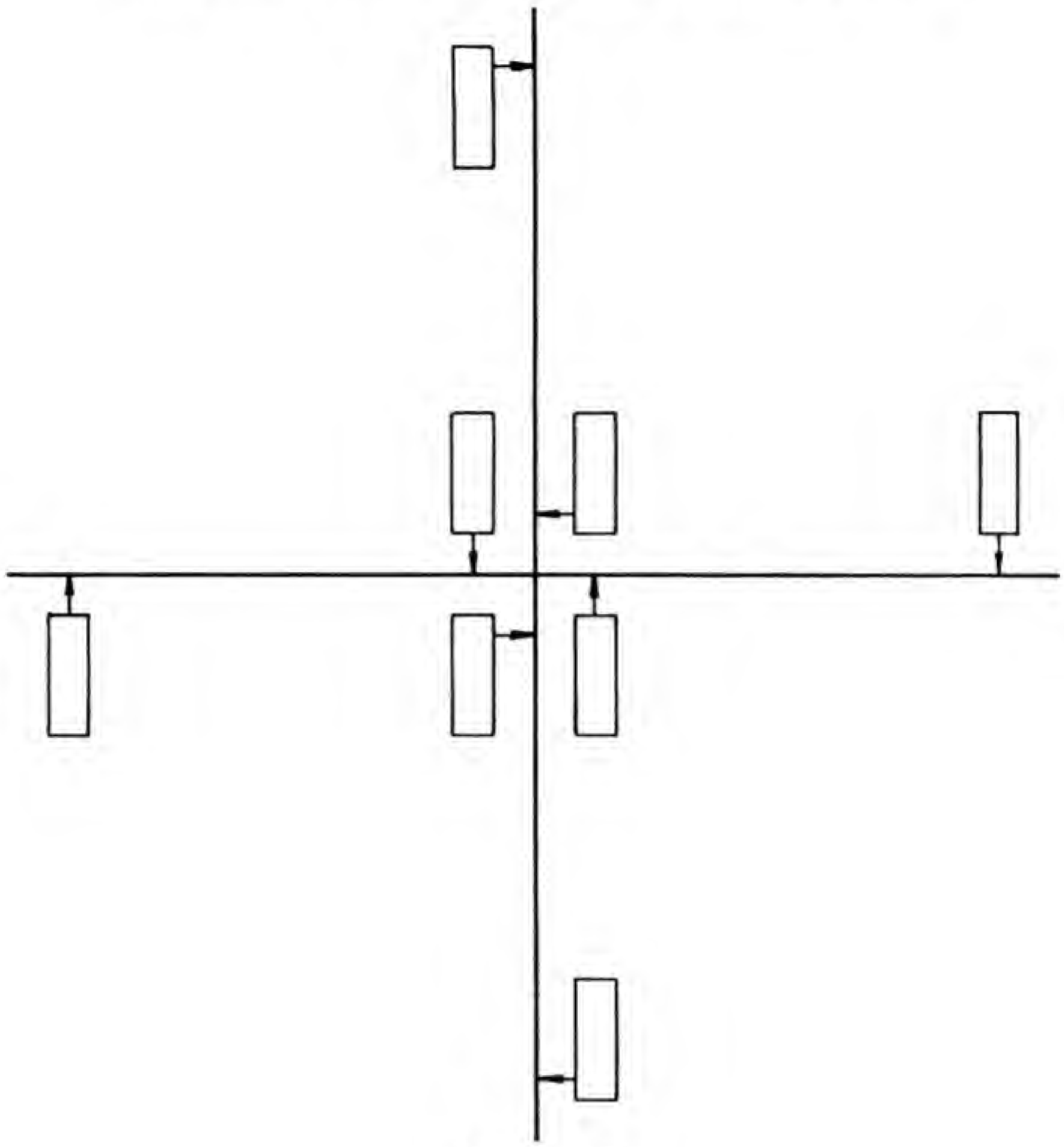


Figure 4.7

EXAMPLE 4.3 *Determine the joint moments of the frame shown in Fig. 4.8.*

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{12}{3} = 4$$

$$K_{BC} = \frac{2 \times 12}{6} = 4$$

$$K_{CD} = \frac{12}{3} = 4$$

$$K_{CE} = \frac{3}{4} \times \frac{1.5 \times 12}{4} = 3.375$$

$$(DF)_{BA} = \frac{4}{4 + 4} = 0.5$$

$$(DF)_{BC} = \frac{4}{4+4} = 0.5$$

$$(DF)_{CB} = \frac{4}{4+4+3.375} = 0.352$$

$$(DF)_{CD} = \frac{4}{11.375} = 0.352$$

$$(DF)_{CE} = \frac{3.375}{11.375} = 0.296$$

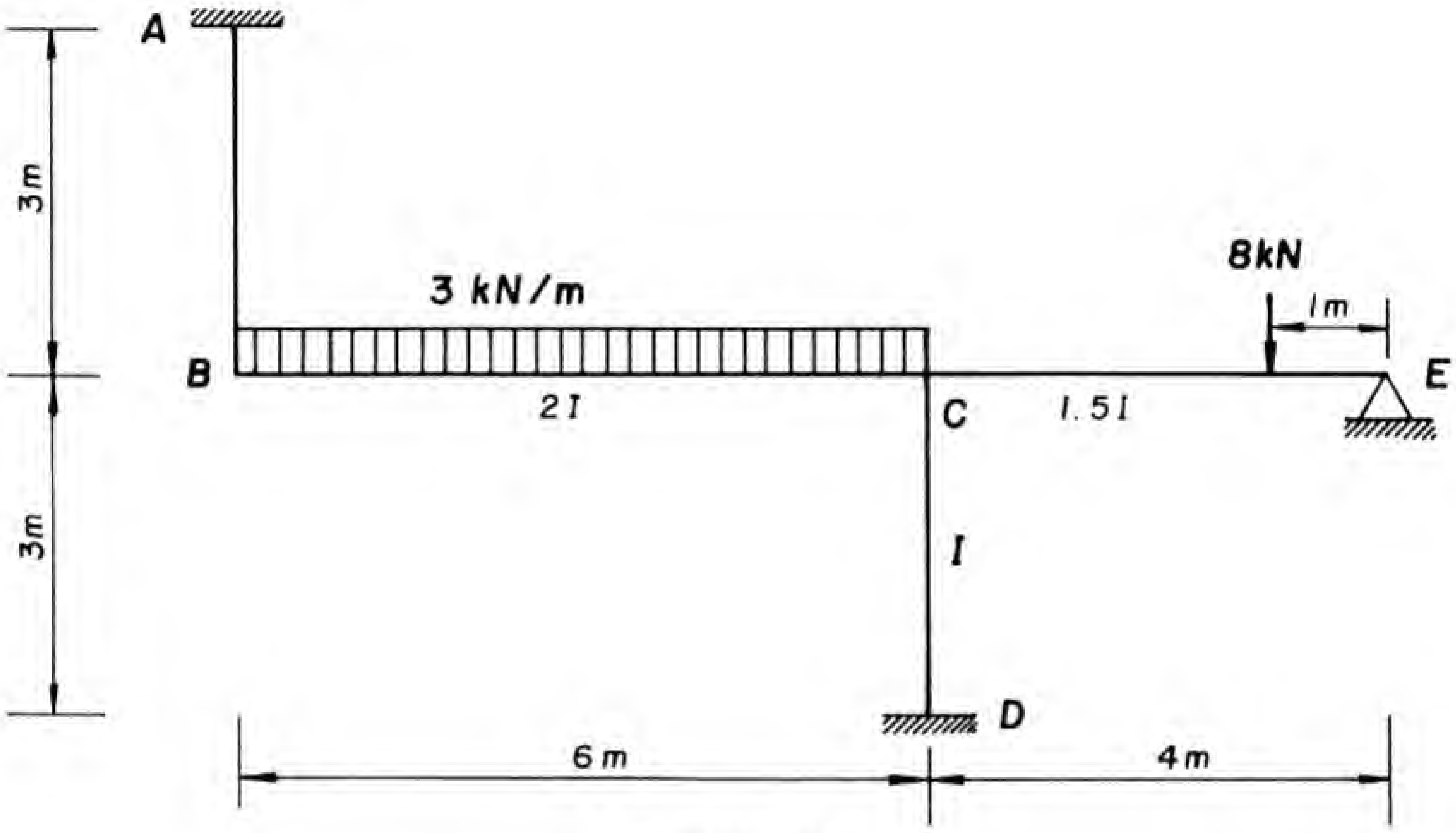


Figure 4.8

Fixed-End Moments

$$M^F_{BC} = -M^F_{CB} = \frac{3(6)^2}{12} = 9 \text{ kN m}$$

$$M^F_{CE} = \frac{8(3)(1)^2}{4^2} = 1.5 \text{ kN m}$$

$$M^F_{EC} = \frac{8(1)(3)^2}{16} = -4.5 \text{ kN m}$$

The moment distribution is performed in tabular form since the number of joints is few.

Table 4.6

Joint	A	B		C			D	E
Member	AB	BA	BC	CB	CE	CD	DC	EC
K	4	4	4	4	3.375	4	4	3.375
DF	0	0.5	0.5	0.352	0.296	0.352	0	
Fixed-end moment			+9.00	−9.00	+1.50 +2.25			−4.50 +4.50
			+0.92	+1.85	+1.55	+1.85	+0.93	
	−2.48	−4.96	−4.96	−2.48				
			+0.44	+0.87	+0.74	+0.87	+0.44	
	−0.11	−0.22	−0.11					
			+0.02	+0.04	+0.03	+0.04	+0.02	
Total	−2.59	−5.19	+5.19	−8.83	+6.07	+2.76	+1.39	0.0

4.6.2 Frames With Sidesway

Rectangular Frames

When a frame is loaded laterally, or in the case when the loading or the frame itself is unsymmetrical, sidesway or joint translations occur. As the joint displacements are unknown the fixed-end moments due to the displacements cannot be calculated. The solution for frames with a single mode of sway is indicated in Fig. 4.9. In applying the moment distribution method the joints are first assumed to be held against sidesway by introducing ***artificial restraint*** R, at the appropriate joints. The fixed-end moments caused by the applied loads are then distributed to obtain the non-sway balanced end moments. Next, the magnitude of the reaction R, at the ***artificial restraint*** is determined from the considerations of equilibrium of the members. The effect of the restraint R is determined by permitting an arbitrary sway to take place, unaccompanied initially by rotation at the joints. These fixed-end moments are then distributed and subsequently, the force F necessary to maintain the frame in its swayed position is determined from the equilibrium condition of the column shears. The actual magnitude of lateral force, F, consistent with the condition necessary to eliminate the ***artificial restraint*** R, and therefore the moments caused by the sidesway, are determined from the condition:

$$R = kF \qquad [4.8]$$

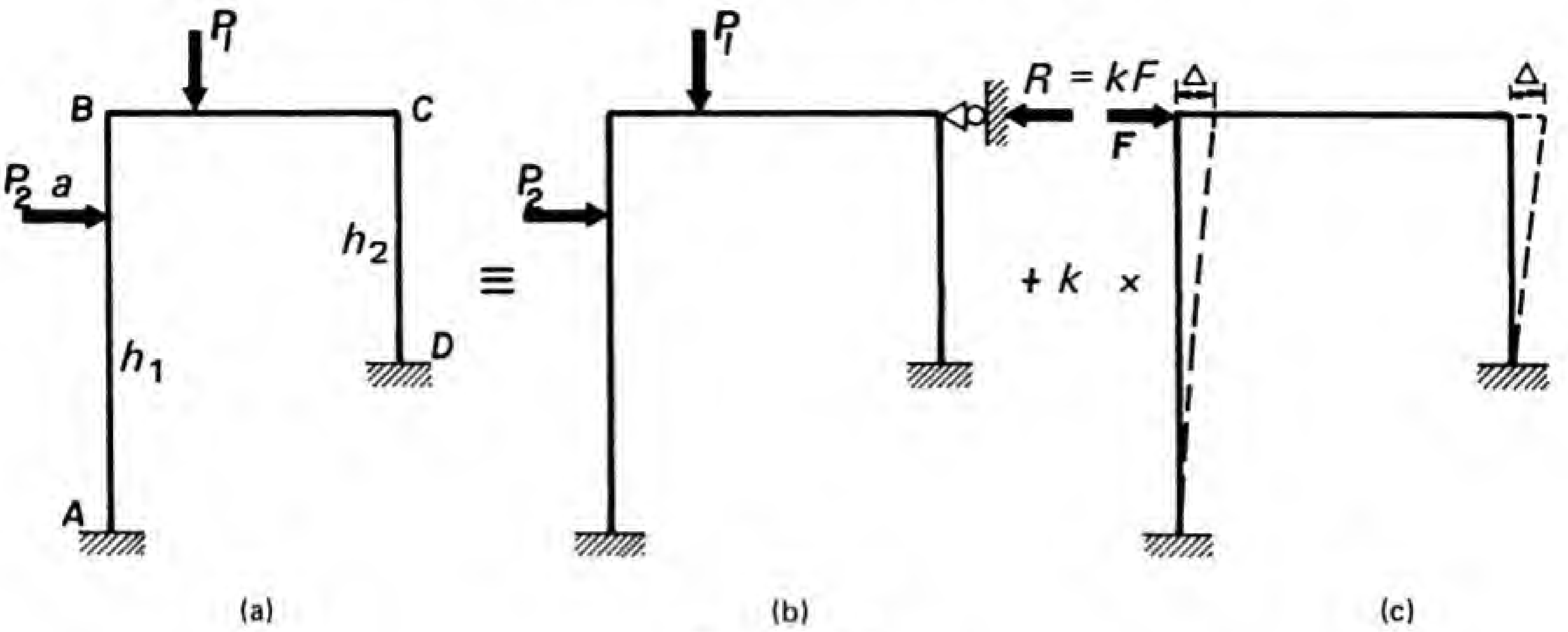


Figure 4.9

The final set of end moments are obtained by adding the original set of moments to the sway moments multiplied by the *correction factor k*. This process is illustrated by the frame shown in Fig. 4.9.

Note that the frame shown in Fig. 4.9(a) is equivalent to the sum of Fig. 4.9(b) and Fig. 4.9(c). In Fig. 4.9(b) the frame is prevented from sidesway by the artificial restraint R applied at joint C. The arbitrary displacement Δ occurs at C' with the joints held against rotation and when the fixed-end moments are distributed the horizontal force that caused the sidesway F is determined from equilibrium conditions. The shear condition of the given frame is

$$H'_A + H'_D - P_2 = R$$

But

$$H'_A = \frac{M'_{AB} + M'_{BA}}{h_1} + \frac{P_2 a}{h_1}$$

$$H'_D = \frac{M_{CD} + M_{DC}}{h_2}$$

Thus

$$\frac{M'_{AB} + M'_{BA}}{h_1} + \frac{P_2 a}{h_1} + \frac{M'_{CD} + M'_{DC}}{h_2} - P_2 = R$$

Similarly

$$H''_A + H''_D = F$$

or

$$\frac{M''_{AB} + M''_{BA}}{h_1} + \frac{M''_{CD} + M''_{DC}}{h_2} = F \qquad [4.9]$$

But

$$R = kF \qquad [4.10]$$

Therefore by superposition condition

$$M_{AB} = M'_{AB} + kM''_{AB}$$
$$M_{BA} = M'_{BA} + kM''_{BA}, \text{ etc.} \qquad [4.11]$$

EXAMPLE 4.4 *Determine the joint moments of the frame shown in Fig. 4.10.*

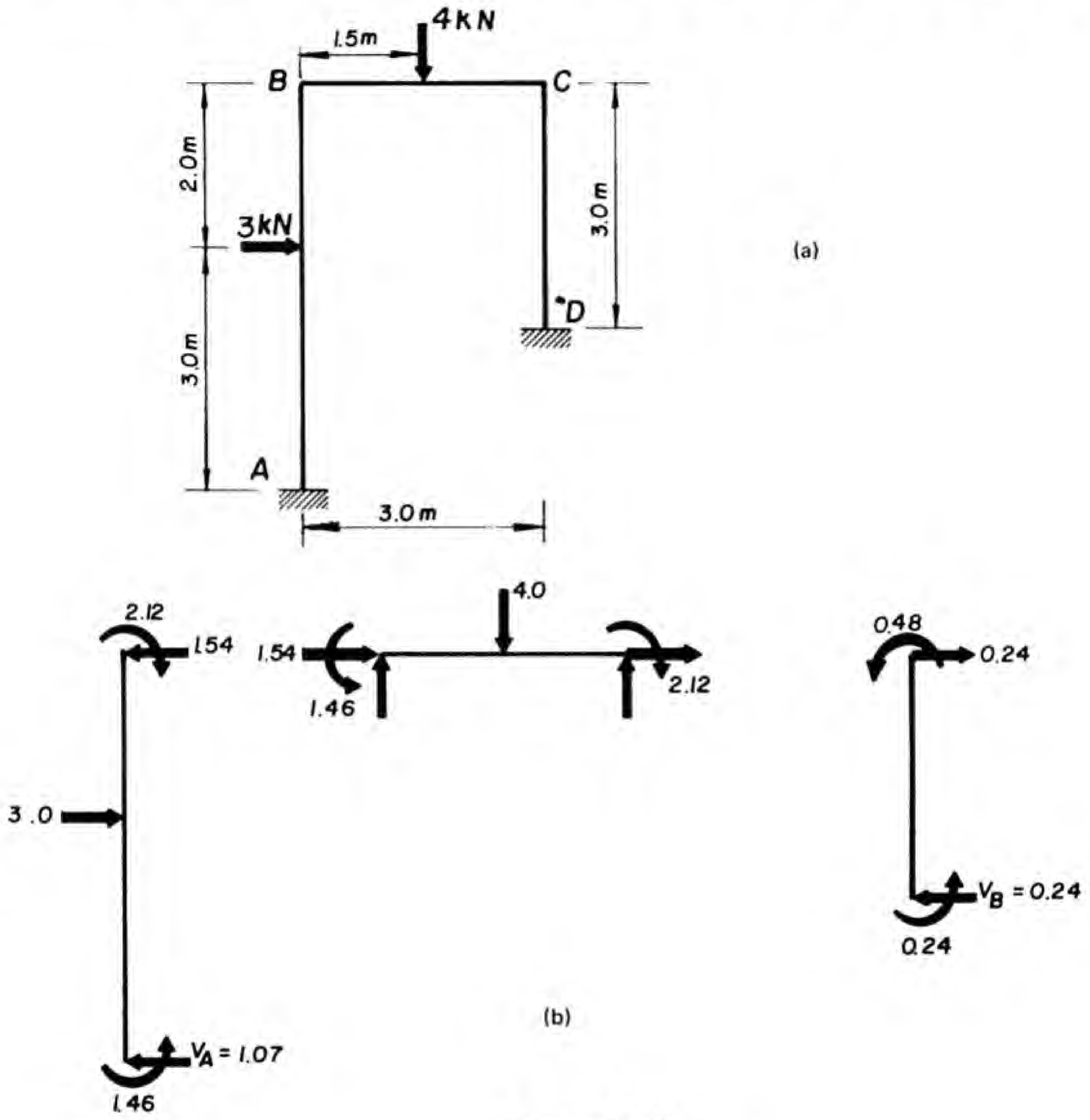


Figure 4.10

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{15}{5} = 3$$

$$K_{BC} = \frac{15 \times 2}{3} = 10$$

$$K_{CD} = \frac{15}{3} = 5$$

$$(DF)_{BA} = \frac{3}{10+3} = 0.231$$

$$(DF)_{BC} = \frac{10}{13} = 0.769$$

$$(DF)_{CB} = \frac{10}{10+5} = 0.667$$

$$(DF)_{CD} = \frac{5}{15} = 0.333$$

Fixed-End Moments

$$M^F_{AB} = \frac{3(3)(2)^2}{3^2} = 1.44 \text{ kN m}$$

$$M^F_{BA} = -\frac{3(2)(3)^2}{25} = -2.16 \text{ kN m}$$

$$M^F_{CB} = -M^F_{BC} = \frac{4(1.5)(1.5)^2}{9} = 1.50 \text{ kN m}$$

(a) Moment Distribution without Sidesway

Table 4.7 Distribution without Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	3	3	10	10	5	5
DF	0.0	0.231	0.769	0.667	0.333	0.0
Fixed-end moment	+1.44	−2.16	+1.5	−1.5		
			+0.5	+1.0	+0.50	+0.25
	+0.02	+0.04	+0.12	+0.06		
				−0.04	−0.02	−0.01
Total	+1.46	−2.12	+2.12	−0.48	+0.48	+0.24

By using the end moments on the free-body diagrams as shown in Fig. 4.10(a) the end shears are determined to be

$$V_A = \frac{3 \times 2 + 1.46 - 2.12}{5} = 1.068 \text{ kN} \quad \text{(left)}$$

$$V_B = \frac{0.48 + 0.24}{3} = 0.240 \text{ kN} \quad \text{(left)}$$

The artificial joint restraint is

$$R = 3.0 - 1.068 - 0.24 = 1.692 \text{ kN} \quad \text{(left)}$$

(b) Moment Distribution with Sidesway

Assume the frame to sway an arbitrary amount Δ such that the fixed-end moments in the columns are

$$M_{AB}^F = -M_{CD}^F = \frac{6EI\Delta}{L^2}$$

$$= \frac{6EI\Delta}{5^2} = 0.24EI$$

$$M_{CD}^F = +M_{DC}^F = \frac{6EI\Delta}{3^2} = 0.667EI$$

Taking $EI = 10$, distribution of the fixed-end moments due to sidesway is shown in Table 4.8.

Table 4.8 Distribution with Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0.0	0.231	0.769	0.667	0.333	0.0
Fixed-end moment	+2.4	+2.4			+6.67	+6.67
	−0.27	−0.55	−1.85	−0.93		
			−1.92	−3.83	−1.91	−0.95
		+0.44	+1.48	+0.74		
			−0.24	−0.49	−0.25	−0.12
	+0.03	+0.06	+0.18			
			+0.03	−0.06	−0.03	
Total	+2.38	+2.36	−2.36	−4.48	+4.48	+5.59

The column shears are

$$V_{AB} = \frac{2.38 + 2.36}{5.0} = 0.948 \text{ kN}$$

$$V_{DC} = \frac{4.48 + 5.59}{3.0} = 3.356 \text{ kN}$$

The net shear is

$$F = V_{AB} + V_{DC}$$

$$= 0.948 + 3.356 = 4.305 \text{ kN}$$

The correction factor is

$$k = \frac{R}{F} = \frac{1.692}{4.305} = +0.393$$

The final end moments are obtained by multiplying the moments from the *sway* solution and then adding the moments from the *non-sway* solution to obtain the moments in the original structure.

Thus,

$$M_{AB} = 1.46 + 2.38(0.393) = +2.40 \text{ kN m}$$

$$M_{BA} = -2.12 + 2.36(0.393) = -1.19 \text{ kN m}$$

$$M_{CB} = -0.48 - 4.48(0.393) = -2.24 \text{ kN m}$$

$$M_{DC} = 0.24 + 5.59(0.393) = 2.44 \text{ kN m}$$

Frames with Inclined Members

The superposition method described for the case of rectangular frames with sidesway, can also be used in the analysis of frames with inclined legs. For example, for the single storey portal frame with inclined legs shown in Fig. 4.11, the frame is held against sidesway by artificial joint restraint R (Fig. 4.11(b)) and, after the moment distribution calculations are carried out, the value of R is determined. In the second analysis, the consistent joint force F is computed due to an arbitrary lateral displacement Δ.

After determining the consistent joint force F, the correction factor k is then obtained to calculate the final moments.

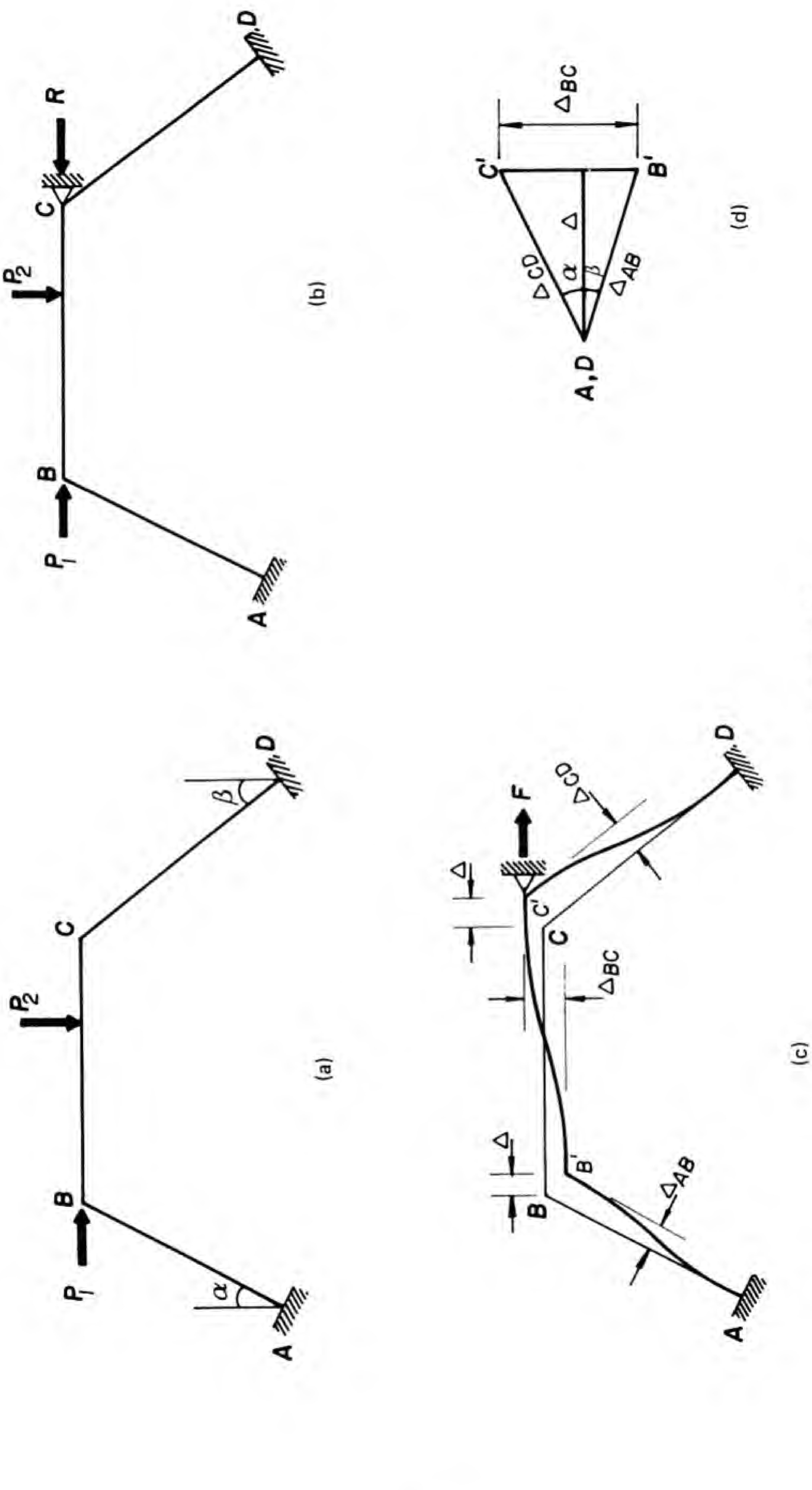

(a)
A
B
C
D
P_1
P_2
α
β
(b)
A
B
C
D
P_1
P_2
R
(c)
A
B
C
D
B'
C'
F
Δ
Δ
Δ_{AB}
Δ_{BC}
Δ_{CD}
(d)
A, D
B'
C'
Δ
Δ_{AB}
Δ_{BC}
Δ_{CD}
α
β

Figure 4.11

EXAMPLE 4.5 *Find the joint moments of the frame shown in Fig. 4.12 by the moment distribution method.*

Relative Stiffness and Distribution Factors

$$K_{AB} = \frac{30}{3} = 10$$

$$K_{BC} = \frac{30 \times 2}{4} = 15$$

$$K_{CD} = \frac{30}{5} = 6$$

$$(DF)_{BA} = \frac{10}{25} = 0.4$$

$$(DF)_{BC} = \frac{15}{25} = 0.6$$

$$(DF)_{CB} = \frac{15}{21} = 0.714$$

$$(DF)_{CD} = \frac{6}{21} = 0.286$$

Fixed-End Moments

$$M^F_{BC} = -M^F_{CB} = \frac{(10)(4)}{8} = 5.0 \text{ kN}$$

(a) Moment Distribution without Sidesway

The member shears are

$$V_{AB} = \frac{1.51 + 3.04}{3} = 1.52 \text{ kN}$$

$$V_{DC} = \frac{1.04 + 2.08}{5} = 0.62 \text{ kN}$$

Figure 4.12(b) shows the forces and reactions on the frame. The artificial

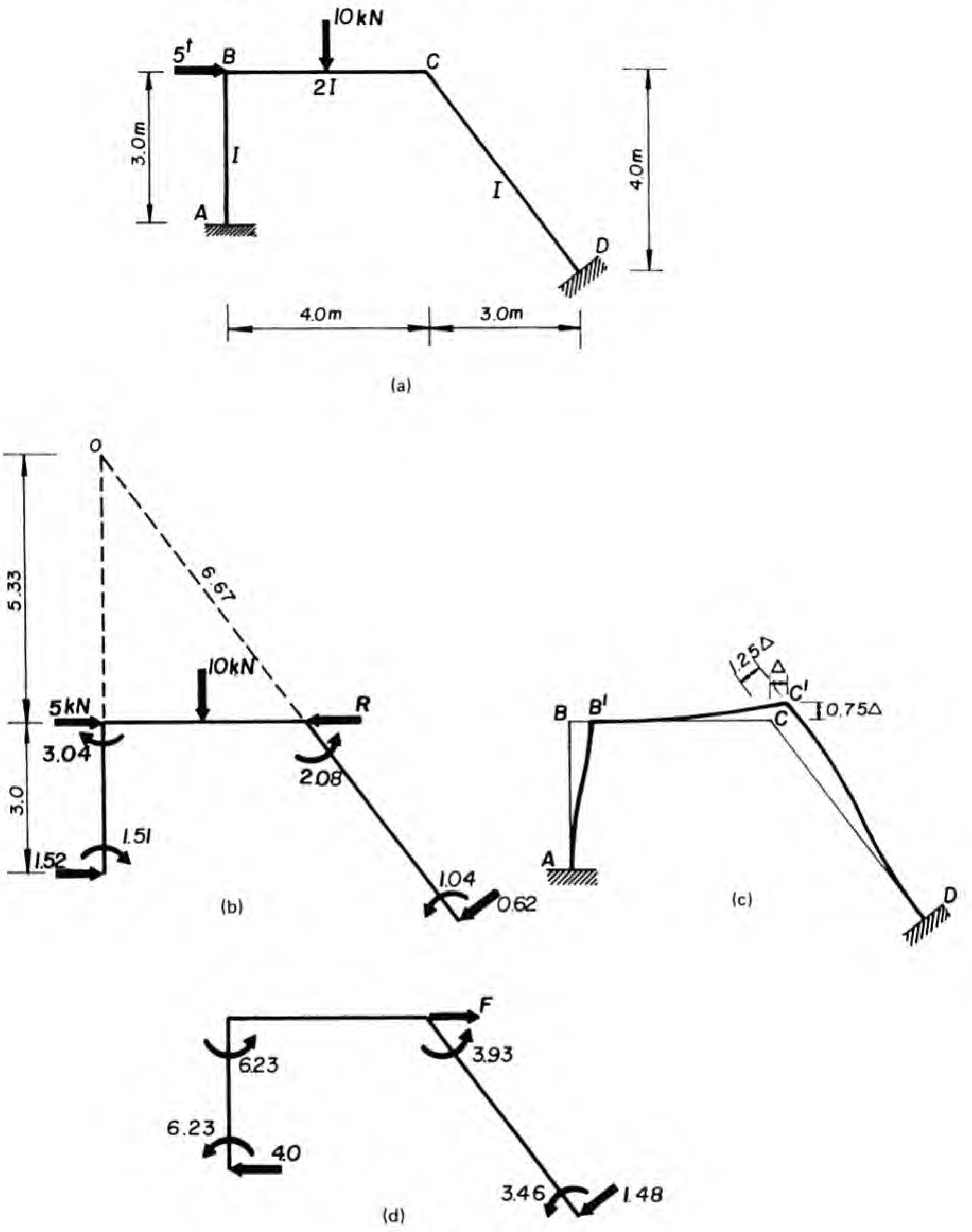


Figure 4.12

joint restraint R may be found by taking moments about the point of intersection of members AB and DC. Thus,

$$5.33R + 10(2) + 0.62(11.67) + 1.51 - 5(5.33) - 1.52(8.33) - 1.02 = 0$$

$$R = 2.17 \text{ kN}$$

Table 4.9 Distribution without Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	10	10	15	15	6	6
DF	0.0	0.40	0.60	0.714	0.286	0.0
Fixed-end			+5.0	−5.0		
moment	−1.0	−2.0	−3.0	−1.5		
			+2.32	+4.64	+1.86	+0.93
	−0.46	−0.93	−1.39	−0.70		
			+0.25	+0.50	+0.20	+0.10
	−0.05	−0.10	−0.15	−0.08		
			+0.03	+0.06	+0.02	+0.01
		−0.01	−0.02			
Total	−1.51	−3.04	+3.04	−2.08	+2.08	+1.04

(b) Moment Distribution with Sidesway

The loads are removed and the frame is permitted to sway to the right through a horizontal distance Δ. As shown in Fig. 4.12(c) the relative displacements are

Member AB : 1.0Δ

Member BC : 0.75Δ

Member CD : 1.25Δ

Fixed-End Moments

$$M_{AB}^{F} = \frac{6EI\Delta}{L^2} = \frac{6EI\Delta}{(3)^2}$$

$$M_{BC}^{F} = -\frac{6E(2I)(0.75\Delta)}{(4)^2}$$

$$M_{CD}^{F} = \frac{6EI(1.25\Delta)}{(5)^2}$$

Assuming $EI\Delta = 10$

$$M^F_{AB} = +6.67 \text{ kN m}$$

$$M^F_{BC} = -5.63 \text{ kN m}$$

$$M^F_{CD} = +3.00 \text{ kN m}$$

Table 4.10 Distribution with Sidesway

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0.0	0.40	0.60	0.714	0.286	0.0
Fixed-end moment	+6.67	+6.67	−5.63	−5.63	+3.00	+3.00
	−0.21	−0.42	−0.62	−0.31		
			+1.05	+2.10	+0.84	+0.42
	−0.21	−0.42	−0.63	−0.31		
			+0.11	+0.22	+0.08	+0.04
	−0.02	−0.05	−0.07	−0.03		
				+0.02	+0.01	
Total	+6.23	+5.78	−5.79	−3.94	+3.93	+3.46

The member shears are

$$V_{AB} = \frac{6.23 + 5.78}{3} = 4.00 \text{ kN} \quad \text{(left)}$$

$$V_{DC} = \frac{3.46 + 3.93}{5} = 1.48 \text{ kN} \quad \text{(left)}$$

The consistent joint force F required to produce the set of moments given in Table 4.10 is found by taking moments about the point of intersection of AB and CD. Thus,

$$5.33F + 6.23 + 3.46 - 4(8.33) - 1.48(11.67) = 0$$

$$F = 7.67 \text{ kN}$$

The correction factor is

$$k = \frac{R}{F} = \frac{2.17}{7.67} = 0.283$$

Final Moments

The final moments are determined by algebraically adding the results of the no-sway solution to the products of k and the corresponding results of the sway solution. Thus

$$M_{AB} = -1.51 + 0.283(+6.23) = +0.25 \text{ kN m}$$

$$M_{BA} = -M_{BC} = -3.04 + 0.283(+5.78) = -1.40 \text{ kN m}$$

$$M_{CB} = -M_{CD} = -2.08 + 0.283(-3.94) = -3.20 \text{ kN m}$$

$$M_{DC} = +1.04 + 0.283(+3.46) = +2.02 \text{ kN m}$$

4.6.3 Frames With Multiple Degrees of Freedom With Respect to Sidesway

A rigid frame which has n independent joint translations is said to have *n degrees of freedom* with respect to sidesway. For example, in the case of the two-storey frame shown in Fig. 4.13(a), with joint B deflecting Δ_B and joint C an independent deflection of Δ_C, the frame is said to have two degrees of freedom with respect to sidesway.

By applying the principle of superposition, the two-storey frame (Fig. 4.13) may be analysed in three separate steps. First, the frame is completely prevented from sidesway by introducing artificial supports as shown in Fig. 4.13(b). In this case all the given loads are applied and a regular no-sway moment distribution is carried out to obtain the artificial joint restraint (assumed positive to the left) R_{10} and R_{20}. In the second step, a translation of the frame of an arbitrary displacement Δ_1 is introduced at joint C (Fig. 4.13(c)). While translation is introduced the joints are locked against rotation and initial moments are developed in members AB and EF as shown by the solid lines. To permit the joints to rotate, shown by the dotted elastic curve, a moment distribution is performed and the consistent joint forces R_{11} and R_{12} are calculated. Finally, a similar solution is conducted for an arbitrary displacement Δ_2 at joint C (Fig. 4.13(d)).

With the separate trial solutions, it now remains to determine how much of the two sidesway solutions should be superimposed to the first case to obtain the final results for the actual given problem. It must be possible to find the final values by obtaining multiplying factors k_1 and k_2 from a linear combination of the obtained results. These factors are obtained by solving two simultaneous equations formulated from the superposition equations for the reactions at D and E:

$$\begin{aligned} R_{10} + k_1R_{11} + k_2R_{12} &= 0 \\ R_{20} + k_1R_{21} + k_2R_{22} &= 0 \end{aligned} \qquad [4.12]$$

After finding the values of the proportionality factor k_1 and k_2, the moments in

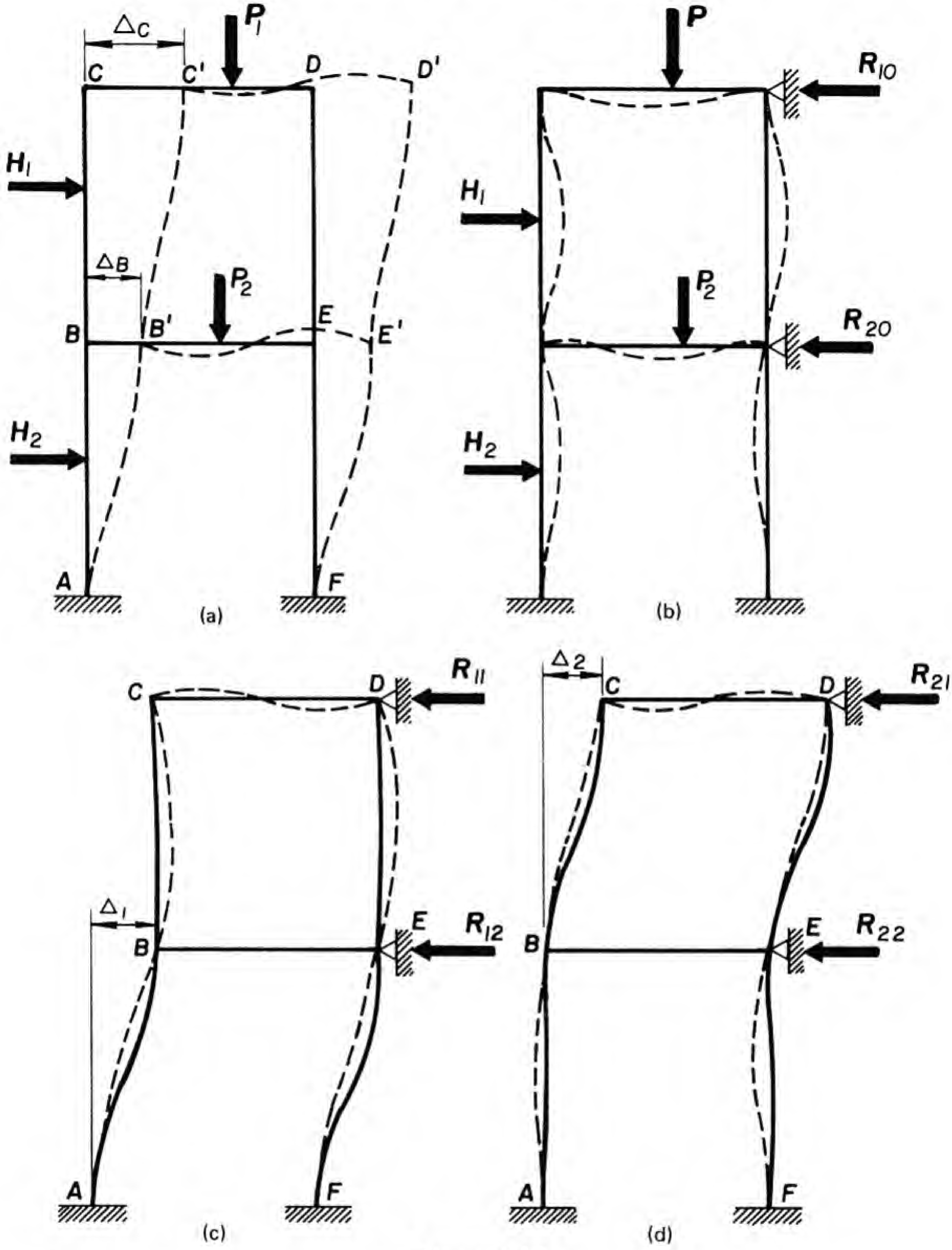


Figure 4.13

the frame are determined from the equation

$$M = M_0 + k_1 M_1 + k_2 M_2 \tag{4.13}$$

where M_0 represents the moments from the non-sway moment distribution, and M_1 and M_2 are the moments due to sidesway, shown in Fig. 4.13(c) and (d) respectively.

The procedure described above can also be used in the analysis of a frame having ***n* degrees of freedom** with respect to sidesway. For such a frame there will be one non-sway moment distribution and **n** separate sidesway cases, in each case only *one* independent sidesway is permitted. After completing the required $(n + 1)$ moment distribution analyses, the **n** superposition equations for the artificial joint restraints and consistent joint are

$$\begin{bmatrix} R_{10} \\ R_{20} \\ \dots \\ R_{30} \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1n} \\ R_{21} & R_{22} & \dots & R_{2n} \\ \dots & \dots & \dots & \dots \\ R_{n1} & R_{n2} & & R_{nn} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \qquad [4.14]$$

or the simultaneous equations may be written in index notation as

$$R_{i0} + \sum_{j=1}^{n} k_i R_{ij} = 0 \quad i = 1, 2, \dots n \qquad [4.15]$$

The solution of the above simultaneous equations gives the values of the multiplying factors $k_1, k_2, \dots k_n$. These values are used to find the final moments.

$$M = M_0 + k_1 M_1 + k_2 M_2 + \dots + k_n M_n$$

$$= M_0 + \sum_{j=1}^{n} k_j M_j \qquad [4.16]$$

Split-Level Frames

The frame of Fig. 4.14 has two degrees of freedom with respect to sidesway. The horizontal displacements of the two roof levels are designated as Δ_B and Δ_D as shown in Fig. 4.14(a). Since Δ_B and Δ_D are unknown, two separate solutions must be obtained where independent sidesways are permitted as shown in Fig. 4.14(c) and (d). With the consistent joint forces R_{10}, R_{12}, R_{21} and R_{22} determined, the proportionality factors k_1 and k_2 are obtained from equilibrium equations as described above.

Gabled Frames

Gabled frames of a single span have two degrees of freedom with respect to joint translation; accordingly, two artificial joint restraints are required to prevent the joints from moving. The steps required for the analysis of gabled frames by moment distribution are shown in Fig. 4.15.

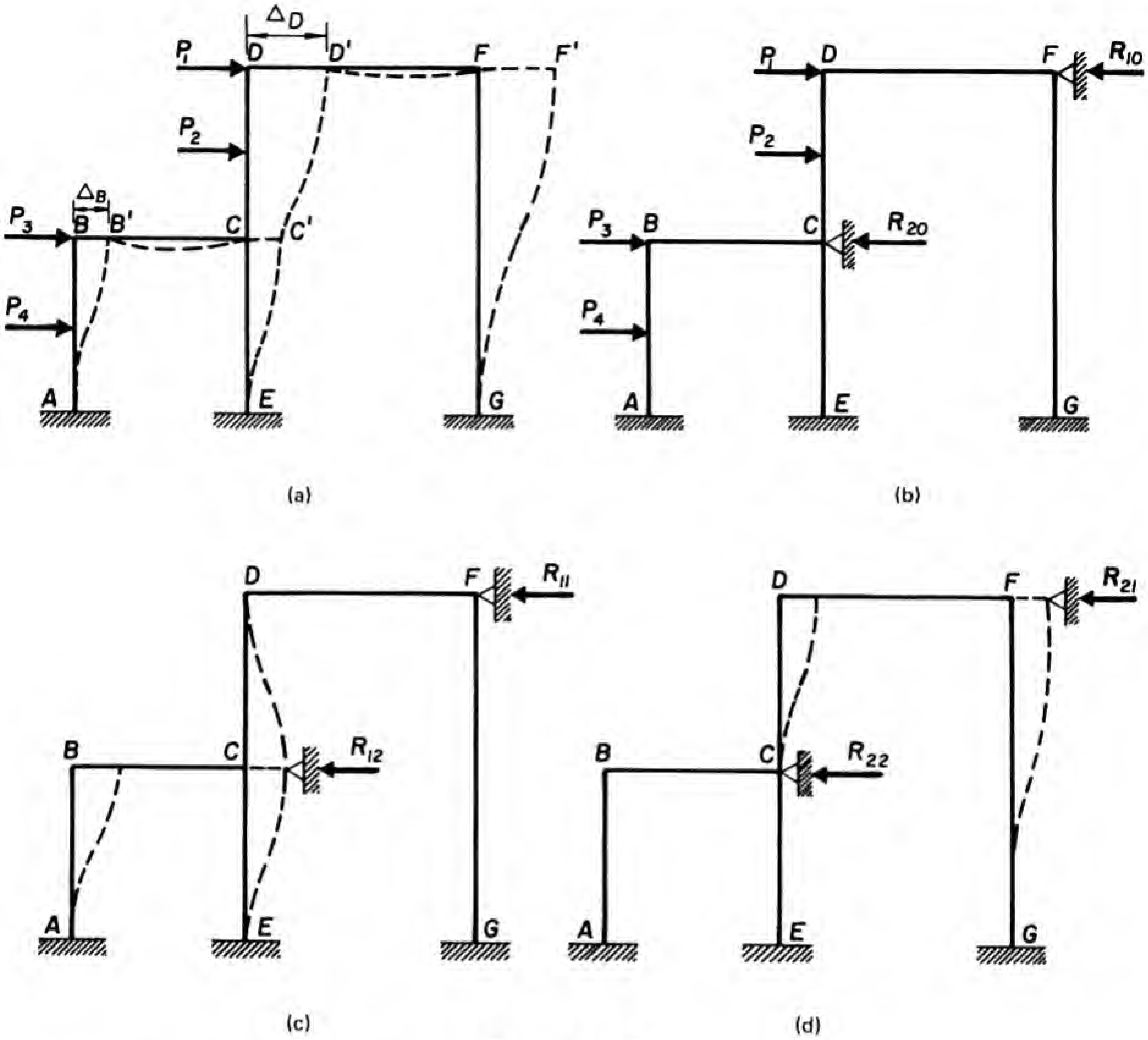


Figure 4.14

4.7 CANTILEVER MOMENT DISTRIBUTION

Unlike the conventional moment distribution, the cantilever moment distribution permits sidesway to occur during the moment balancing process. The method, therefore, makes it possible to evaluate the moments in one distribution without requiring artificial joint restraints. Within its range of applicability the cantilever moment distribution provides a simple but powerful method of analysing *symmetric frames* subjected to lateral loads at storey heights or to *antisymmetric loads*. The derivation of the basic equations necessary for the development of the cantilever moment distribution is given below.

(a) Stiffness Factor of a Cantilever

Consider a uniform cantilever beam AB subjected to an end moment M_{AB} as shown in Fig. 4.16.

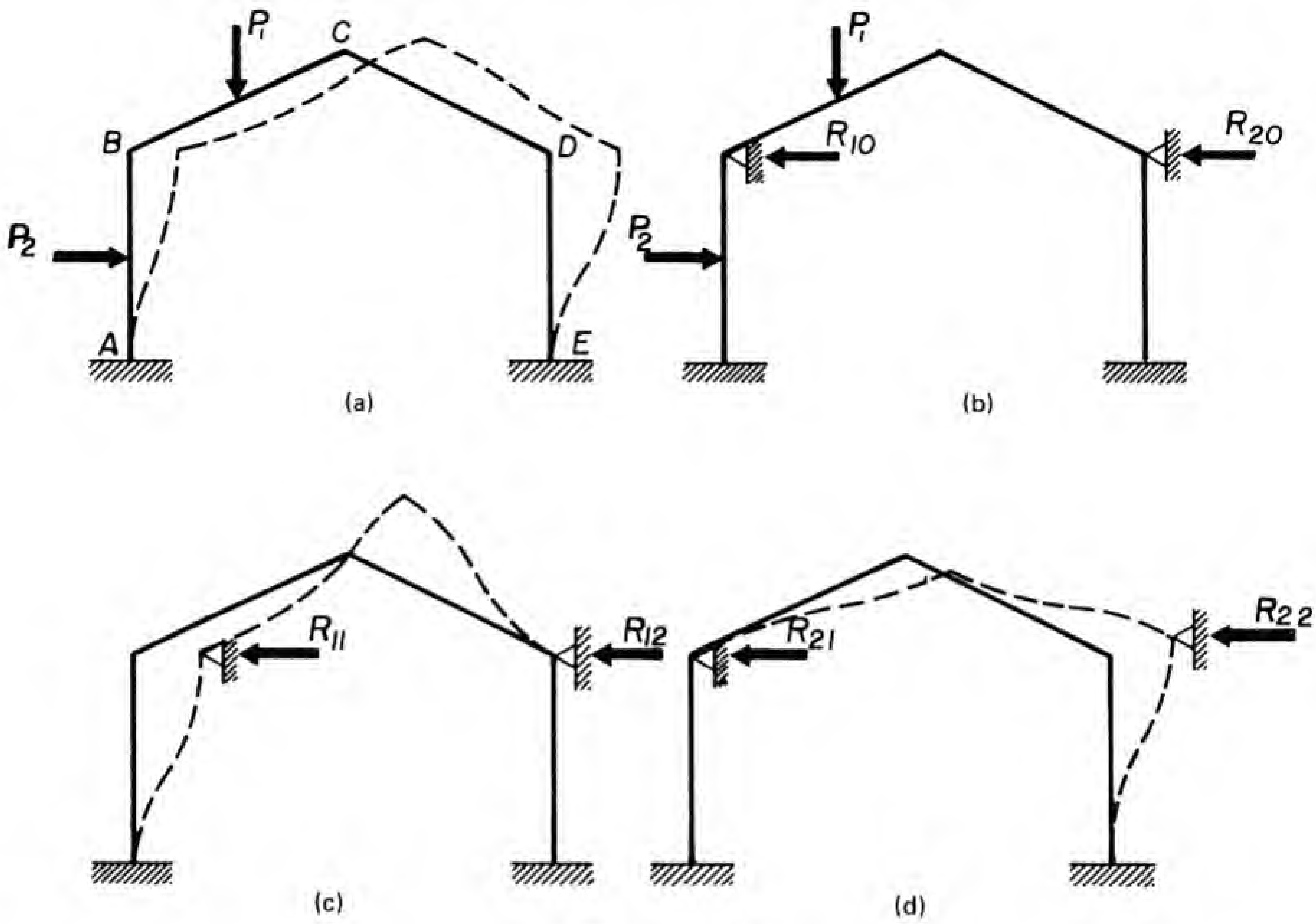


Figure 4.15

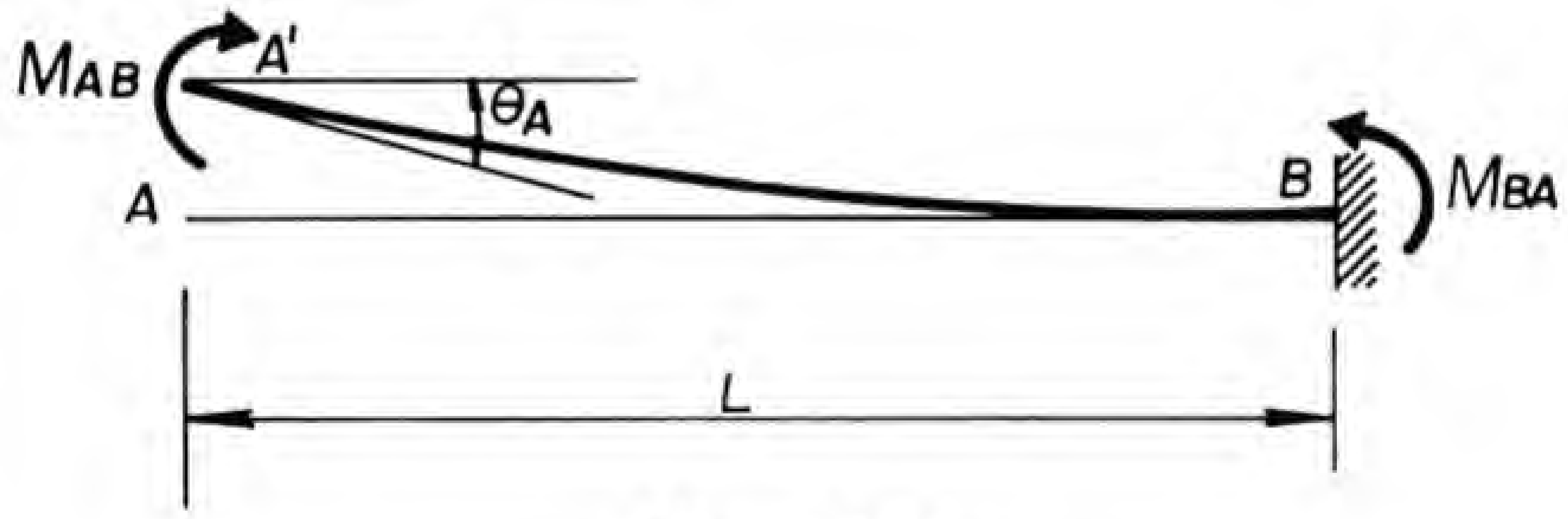


Figure 4.16

It is seen that the angle of rotation is given by the expression

$$\theta_{AB} = \frac{M_{AB}L}{EI}$$

or [4.17]

$$\frac{M_{AB}}{\theta_{AB}} = \frac{EI}{L}$$

which is defined as the *rotation stiffness* of the cantilever beam. Notice that the stiffness of beam AB is one-fourth of the same beam with support A hinged.

(b) Carry-over Factor

The bending moment at end B of the beam (Fig. 4.16), in accordance with the statical sign convention, is equal in magnitude but opposite in direction to the applied moment at support A. The translational carry-over factor is therefore -1.0.

(c) Stiffness Factor of Beams under Antisymmetric Bending

Consider a uniform beam (Fig. 4.17) under antisymmetric bending moments at the ends.

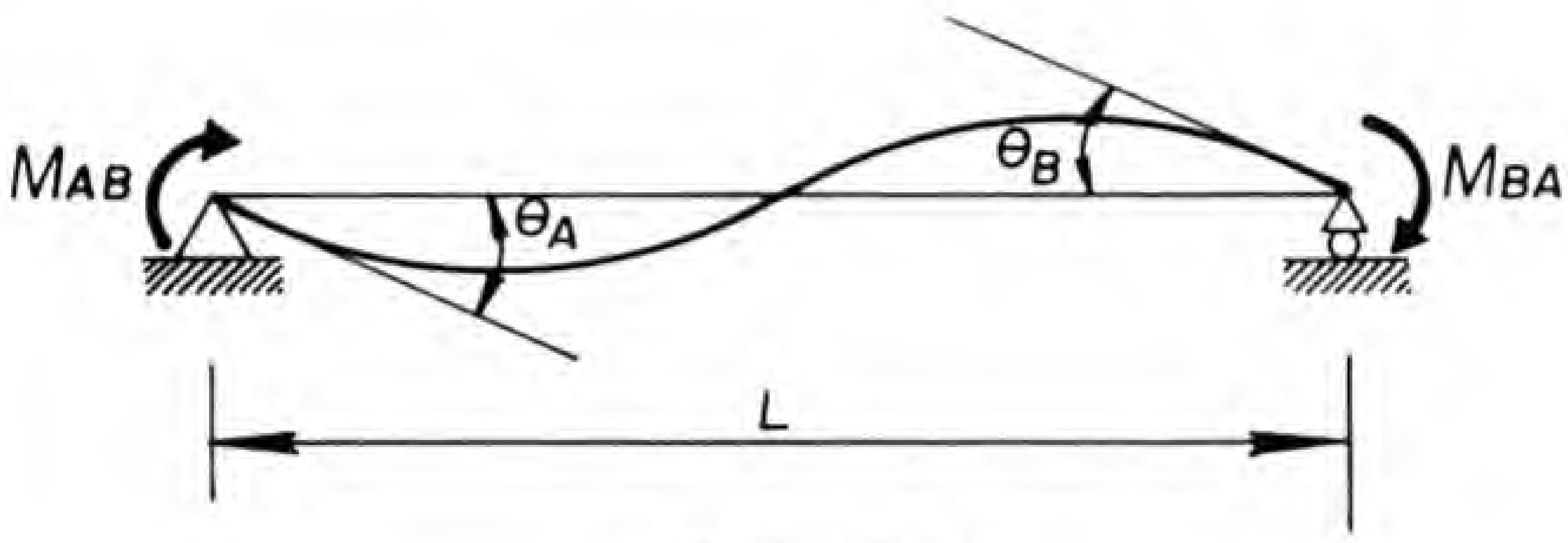


Figure 4.17

The slope-deflection equation for the beam is

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B)$$

or [4.18]

$$\frac{M_{AB}}{L} = \frac{6EI}{L}$$

which is the *rotation stiffness* of a simply supported beam under antisymmetric end moments.

The steps in applying the cantilever moment distribution method may be summarised as follows:

(a) Evaluate the stiffness value for a member parallel to the axis of symmetry from EI/L. The carry-over factor is -1.0 when the far end of the member is fixed.
(b) The stiffness factor for a member perpendicular to the axis of symmetry is $6EI/L$.
(c) Compute the fixed-end moments from the condition that the joints are locked against rotation but free to translate. For more than one-storey

frames, the fixed-end moments are computed to be directly proportional to the product of the storey-shear and storey-height.

(d) Balance the moments for one-half of the structure.
(e) Determine the correction factor to satisfy horizontal equilibrium condition.
(f) Compute final joint moments by multiplying with the correction factor the moments obtained in step (d).

EXAMPLE 4.6 *Determine the joint moments of the frame in Fig. 4.18 using (a) the conventional moment distribution method (b) the cantilever moment distribution method.*

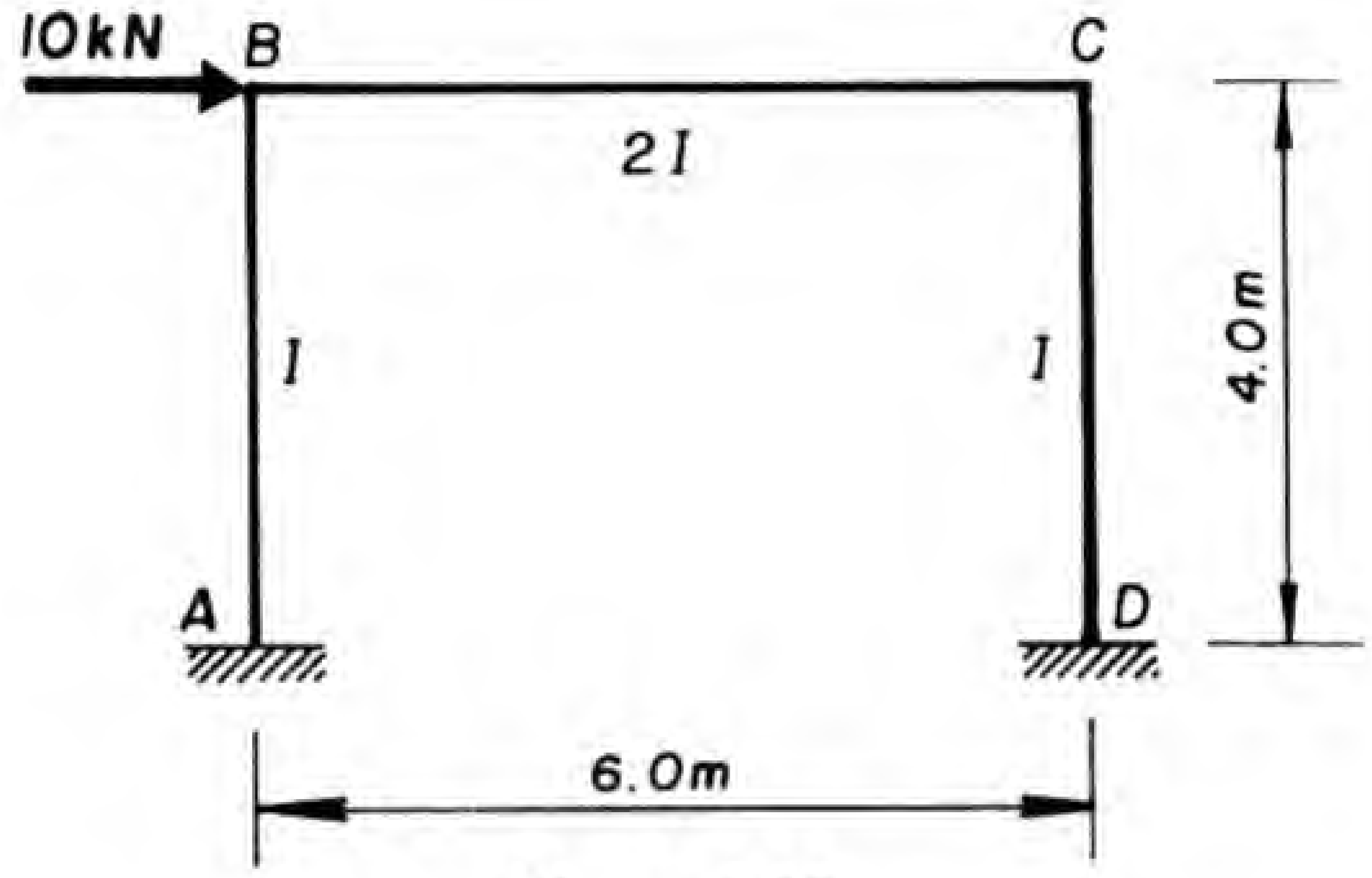


Figure 4.18

(a) Conventional Moment Distribution Method

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{CD} = \frac{1}{4} = 0.25$$

$$K_{BC} = \frac{2}{6} = 0.33$$

$$(DF)_{BA} = (DF)_{CD} = \frac{0.25}{0.25 + 0.33} = 0.431$$

$$(DF)_{BC} = (DF)_{CB} = \frac{0.33}{0.58} = 0.569$$

Fixed-End Moments (Relative Values)

$$M_{AB}^F = M_{BA}^F = M_{CD}^F = M_{DC}^F = \frac{6EI\Delta}{L^2} = 10.0 \text{ kN m}$$

Table 4.11

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
K	0.25	0.25	0.33	0.25	0.25	0.25
DF	0.0	0.431	0.569	0.569	0.431	
Fixed-end moment	+10.0	+10.0			+10.0	+10.0
			−2.85	−5.69	−4.31	−2.15
	−1.54	−3.08	−4.07	−2.04		
			+0.58	+1.16	+0.88	+0.44
	−0.13	−0.25	−0.33	−0.16		
			+0.04	+0.09	+0.07	+0.03
	−0.01	−0.02	−0.02			
Total	+8.32	+6.65	−6.65	−6.64	+6.64	8.32

The base shear is

$$V = \frac{1}{4.0}(M_{AB} + M_{BA} + M_{CD} + M_{DC})$$

$$= \frac{2}{4.0}(8.32 + 6.65) = 7.485 \text{ kN}$$

The correction factor is

$$k = \frac{10.0}{7.485} = 1.336$$

The final end moments are

$$M_{AB} = M_{DC} = 1.336 \times 8.32 = 11.12 \text{ kN m}$$

$$M_{BA} = M_{CD} = 1.336 \times 6.65 = 8.88 \text{ kN m}$$

(b) Cantilever Moment Distribution Method

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{CD} = \frac{EI}{L} = \frac{1}{4.0} = 0.25$$

$$K_{CB} = \frac{6EI}{L} = \frac{6 \times 2}{6.0} = 2.0$$

$$(DF)_{BA} = \frac{2.0}{2.0 + 0.25} = 0.11$$

$$(DF)_{BC} = \frac{2}{2.25} = 0.89$$

Fixed-End Moments (Relative Values)

$$M^F_{AB} = M^F_{BA} = \frac{6EI}{L^2} = 10.0 \text{ kN m}$$

The distribution is carried out in tabular form as shown in Table 4.12. Notice that the fixity at joint B does not exist, it is therefore released by applying at joint B a balancing moment of -10.0 kN m. The moment at joint B is distributed as $-10.0 \times 0.89 = -8.9$ kN m to member BC and $-10.0 \times 0.11 = -1.1$ kN m to member BA. The carry-over factor for column BA being -1.0, the moment carried over to joint A from B is $+1.1$ kN m.

Table 4.12

Joint	A	B	
Member	AB	BA	BC
DF	0.0	0.11	0.89
Fixed-end moment	+10.0 +1.1	+10.0 −1.1	 −8.9
Total	−11.1	+8.9	−8.9

The base shear is

$$V = \frac{2}{4}(11.1 + 8.9) = 10.0 \text{ kN m}$$

The correction factor is

$$k = \frac{10.0}{10.0} = 1.0$$

The final joint moments are

$$M_{AB} = M_{DC} = 1.0(11.1) = 11.1 \text{ kN m}$$

$$M_{BA} = M_{CD} = 1.0(8.9) = 8.9 \text{ kN m}$$

It is noted that the computations are carried out on one-half of the frame only since the moments on either sides of the axis of symmetry of the frame are identical.

EXAMPLE 4.7 *Find the joint moments of the two-storey frame shown in Fig. 4.19 using the cantilever moment distribution method.*

Relative Stiffness Values and Distribution Factors

$$K_{AB} = K_{EF} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = K_{DE} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{6(2I)}{4} = 3.0I$$

$$(DF)_{BA} = \frac{0.50}{0.50 + 0.33 + 3.0} = 0.130$$

$$(DF)_{BE} = \frac{3.0}{3.83} = 0.783$$

$$(DF)_{BC} = \frac{0.33}{0.83} = 0.087$$

$$(DF)_{CB} = \frac{0.33}{0.33 + 3.0} = 0.10$$

$$(DF)_{CD} = \frac{3}{3.33} = 0.90$$

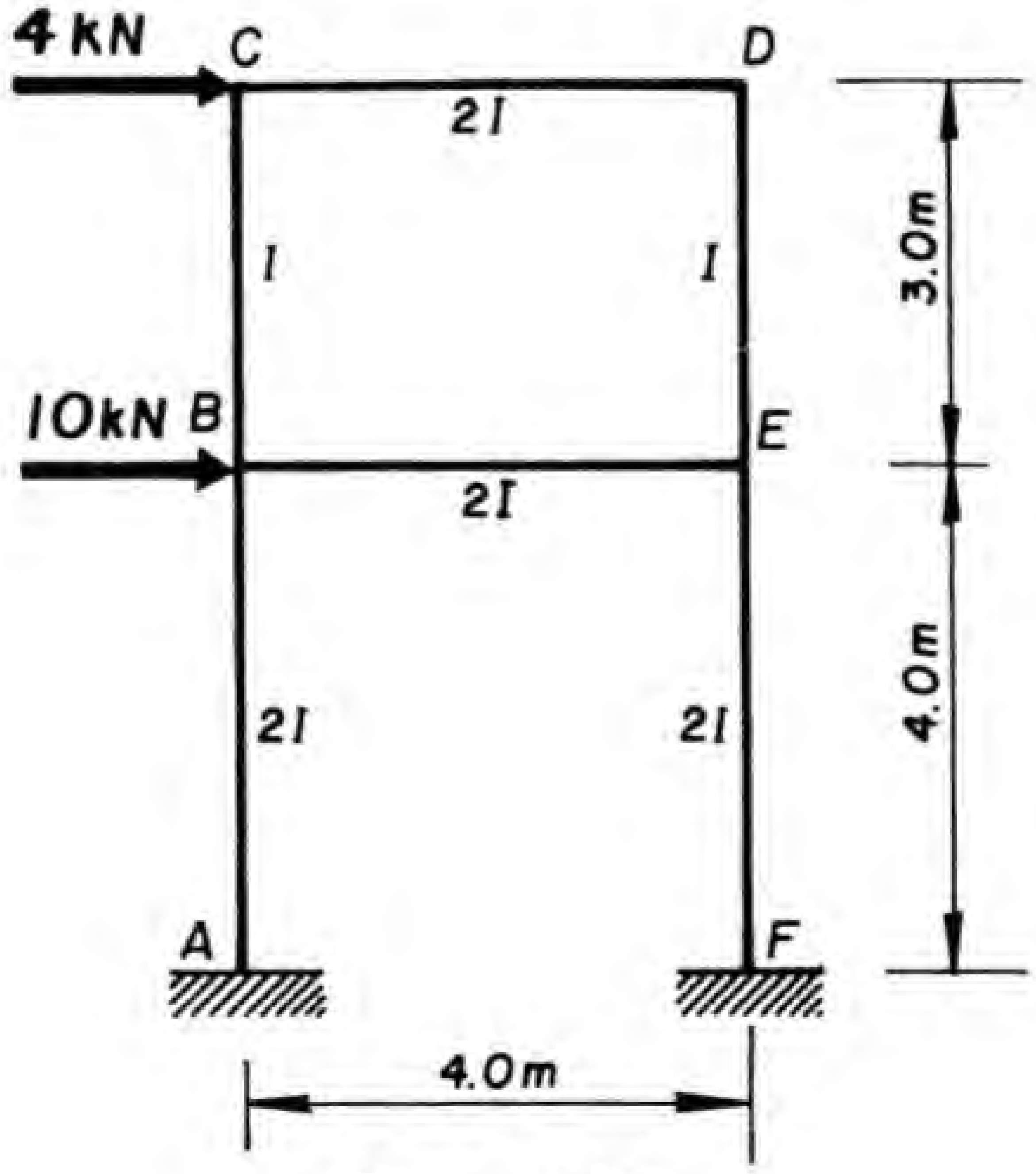


Figure 4.19

Fixed-End Moments (Relative Values)

The fixed-end moments are taken as the product of the storey-shear and the storey-height since the columns behave as a cantilever. Therefore,

$$M_{BC}^F = M^F$$

Subsequently, in the upper storey the fixed-end moments are found by multiplying the storey-shear (4.0 kN) by the column height (3.0 m). In the lower storey the total horizontal force is 14.0 kN.

Thus

$$M_{BC}^F = M_{CB}^F = M_{ED}^F = M_{DE}^F = 4 \times 3 = 12 \text{ kN m}$$

$$M_{AB}^F = M_{BA}^F = M_{EF}^F = M_{FE}^F = 14 \times 4 = 56 \text{ kN m}$$

Table 4.13

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	3.0	0.33	0.33	3.0
DF	0.0	0.130	0.783	0.087	0.10	0.90
Fixed-end moment	+56.00	+56.00		+12.00	+12.00	
	+8.84	−8.84	−53.24	−5.92	+5.92	
				+1.79	−1.79	−16.13
	+0.23	−0.23	−1.40	−0.16	+0.16	
				+0.02	−0.02	
		−0.00	−0.02	−0.00		
Total	+65.07	+46.93	−54.66	+7.73	+16.27	−16.27

Correction Factor

Upper storey shear

$$V_1 = \frac{1}{3}(M_{CB} + M_{BC} + M_{DE} + M_{ED})$$

$$= \frac{2}{3}(7.73 + 16.27) = 16 \text{ kN}$$

Lower storey shear

$$F_2 = \frac{2}{4}(65.07 + 46.93) = 56 \text{ kN}$$

A correction factor must be applied at each level to satisfy horizontal equilibrium.

Since the relative fixed-end moments are taken as the product of storey-shear and storey-height, the correction factor becomes common to both storeys and is given as

$$\text{correction factor} = \frac{4}{16} = \frac{14}{56} = 0.25$$

The final end moments are obtained by multiplying those obtained in Table 4.13 with the correction factor.

Final End Moments

$$M_{AB} = 0.25(+65.07) = +16.27 \text{ kN m}$$

$$M_{BA} = 0.25(+46.93) = +11.73 \text{ kN m}$$

$$M_{BE} = 0.25(-54.66) = -13.67 \text{ kN m}$$

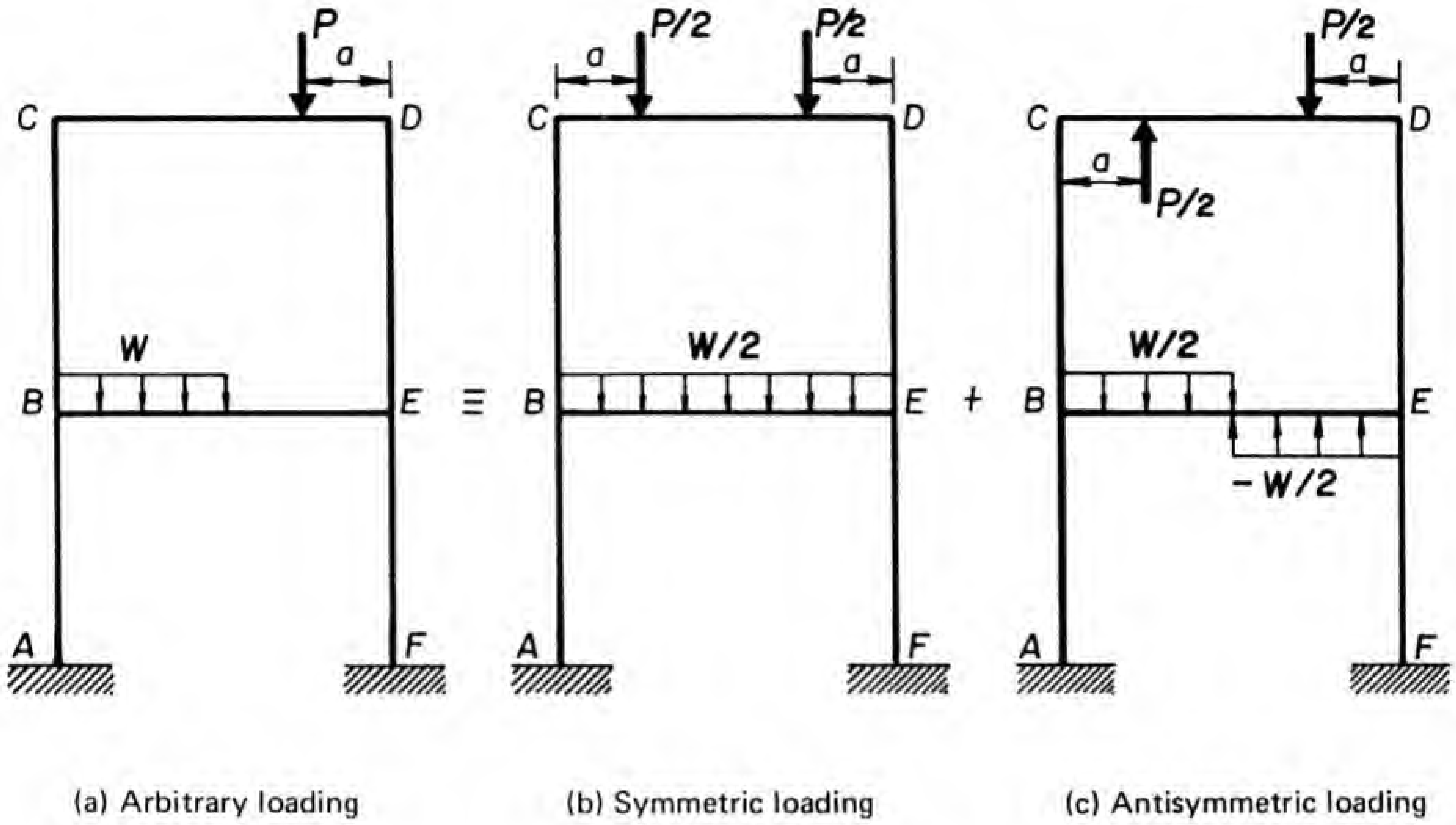
$$M_{BC} = 0.25(+7.73) = +1.93 \text{ kN m}$$

$$M_{CB} = 0.25(+16.27) = +4.07 \text{ kN m}$$

$$M_{CD} = 0.25(-16.27) = -4.07 \text{ kN m}$$

4.8 ARBITRARY LOADING ON SYMMETRIC FRAMES

Any arbitrary loading on a symmetric frame can be *resolved* into *symmetric* and *antisymmetric* loading system components. The frame may then be analysed by using the conventional moment distribution method to the symmetric loading system, and the cantilever moment distribution method for the frame subjected to the antisymmetric loading. The final end moments are then obtained by adding algebraically the results of the two solutions. Figure 4.20 shows a symmetric frame subjected to any arbitrary loading. The same frame is also shown loaded by symmetric and antisymmetric (Fig. 4.20(b) and (c)) loading systems whose algebraic sum furnishes an equivalent system to the original loading.



(a) Arbitrary loading (b) Symmetric loading (c) Antisymmetric loading

Figure 4.20

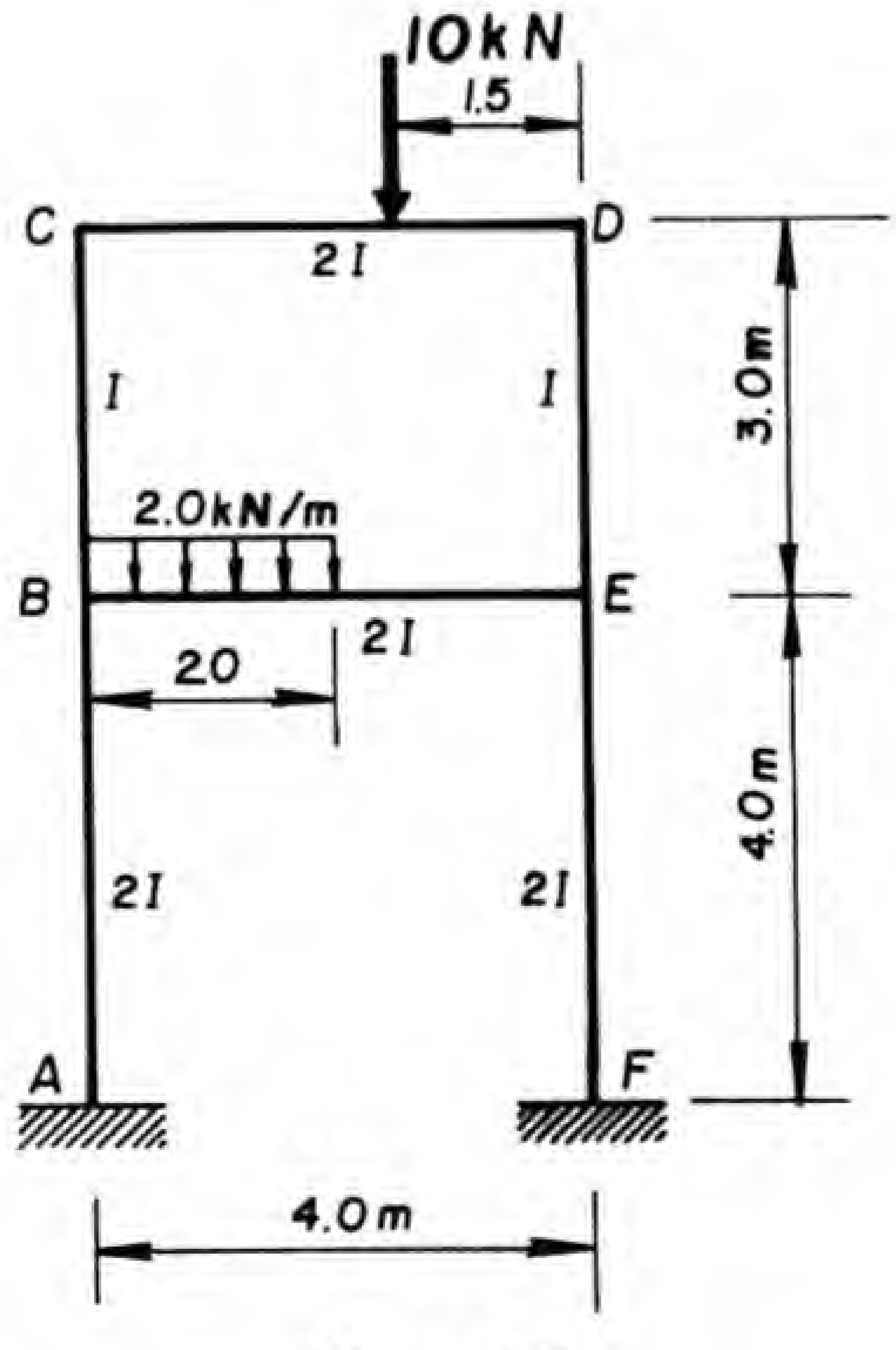


Figure 4.21

EXAMPLE 4.8 *Determine the joint moments of the two-storey frame in Fig. 4.21.*

(a) Symmetric Loading

The frame is analysed with the conventional moment distribution method on half of the frame only, since the frame and the loading are both symmetrical.

Relative Stiffness and Distribution Factor

$$K_{AB} = K_{EF} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = K_{DE} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{2I}{4} = 0.50I$$

$$(DF)_{BA} = (DF)_{BE} = \frac{0.50}{0.50 + 0.33 + 0.50} = 0.375$$

$$(DF)_{BC} = \frac{0.333}{1.333} = 0.25$$

$$(DF)_{CB} = \frac{0.333}{0.333 + 0.50} = 0.40$$

$$(DF)_{CD} = \frac{0.50}{0.833} = 0.60$$

Fixed-End Moments

$$M^F_{CD} = -M^F_{DC} = \frac{5(2.5)(1.5)^2}{(4.0)^2} + \frac{5(1.5)(2.5)^2}{(4.0)^2} = 4.69 \text{ kN m}$$

$$M^F_{BE} = -M^F_{ED} = \frac{1.0(4.0)^2}{12} = 1.33 \text{ kN m}$$

Table 4.14 shows the distribution of the fixed-end moments carried over the left half of the frame. The moments carried from the right half are indicated in *italics*.

Table 4.14

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	0.50	0.333	0.333	0.50
DF	0.0	0.375	0.375	0.250	0.40	0.60
Fixed-end			+1.33			+4.69
moment	−0.25	−0.50	−0.50	−0.33	−0.17	
			+0.25	−0.90	−1.81	−2.71
	+0.12	+0.24	+0.24	+0.17	+0.08	*+1.35*
			−0.12	−0.29	−0.57	−0.86
	+0.07	+0.14	+0.14	+0.09	+0.04	*+0.43*
			−0.03	−0.09	−0.19	−0.28
	+0.03	+0.06	+0.06	+0.05	+0.02	*+0.14*
			−0.03	−0.03	−0.06	−0.08
	+0.01	+0.02	+0.02	+0.02		
Total	−0.02	−0.04	+1.32	−1.27	−2.66	−2.66

(b) Antisymmetric Loading

The cantilever moment distribution method is used to analyse the frame.

Relative Stiffnesses and Distribution Factors

$$K_{AB} = \frac{2I}{4} = 0.50I$$

$$K_{BC} = \frac{I}{3} = 0.33I$$

$$K_{BE} = K_{CD} = \frac{6(2I)}{4} = 3.0I$$

$$(DF)_{BA} = \frac{0.50}{0.50 + 0.33 + 3.0} = 0.130$$

$$(DF)_{BC} = \frac{0.33}{0.83} = 0.087$$

$$(DF)_{BE} = \frac{3.0}{3.83} = 0.783$$

$$(DF)_{CB} = \frac{0.33}{0.33 + 3.0} = 0.10$$

$$(DF)_{BC} = \frac{3.0}{3.33} = 0.90$$

Fixed-End Moments

The fixed-end moment for a beam with a uniformly distributed load extended kL from the left end (support A) is

$$M^F_{AB} = wL^2 \left(\frac{k^2}{12}\right)(6 - 8k + 3k^2)$$

$$M^F_{BA} = -wL^2 \left(\frac{k^3}{12}\right)(4 - 3k)$$

For the problem at hand, $k = 0.5$. Thus

$$M^F_{BE} = 1.0(4.0)^2 \frac{(0.5)^2}{12} [6 - 8(0.5) + 3(0.5)^2]$$

$$- 1.0(4.0)^2 \frac{(0.5)^3}{12} (4 - 3(0.5))$$

$$= +0.500 \text{ kN m}$$

$$M^F_{CD} = -\frac{5(1.5)(2.5)^2}{(4.0)^2} + \frac{5(2.5)(1.5)^2}{(4.0)^2}$$

$$= -1.172 \text{ kN m}$$

These are distributed using the cantilever moment distribution method as shown in Table 4.15.

Final End Moments

$M_{AB} = -0.02 + 0.050 = +0.030$ kN m $\quad M_{DC} = +2.66 - 0.156 = -2.504$ kN m

$M_{BA} = -0.04 - 0.50 = -0.090$ kN m $\quad M_{DE} = +2.66 + 0.146 = +2.806$ kN m

$M_{BC} = -1.27 - 0.146 = -1.416$ kN m $\quad M_{EB} = -1.32 + 0.196 = -1.124$ kN m

$M_{BE} = +1.32 + 0.196 = +1.516$ kN m $\quad M_{ED} = +1.27 - 0.146 = +1.124$ kN m

$M_{CB} = -2.66 + 0.146 = -2.514$ kN m $\quad M_{EF} = +0.04 - 0.050 = -0.10$ kN m

$M_{CD} = -2.66 - 0.156 = -2.816$ kN m $\quad M_{FE} = +0.02 + 0.050 = +0.070$ kN m

Table 4.15

Joint	A	B			C	
Member	AB	BA	BE	BC	CB	CD
K	0.50	0.50	3.0	0.33	0.33	3.0
DF	0.0	0.130	0.783	0.087	0.10	0.90
			+0.500			−1.172
	+0.065	−0.065	−0.392	−0.043	+0.043	
				−0.113	+0.113	+1.016
	−0.015	+0.015	+0.088	+0.010	−0.10	
Total	+0.050	−0.050	+0.196	−0.146	+0.146	−0.156

4.9 PROBLEMS

4.1 Find the support moments of the continuous beam shown in Fig. P4.1.

(Ans: $M_A = -12.6$ kN m

$M_B = -14.4$ kN m

$M_C = -2.2$ kN m)

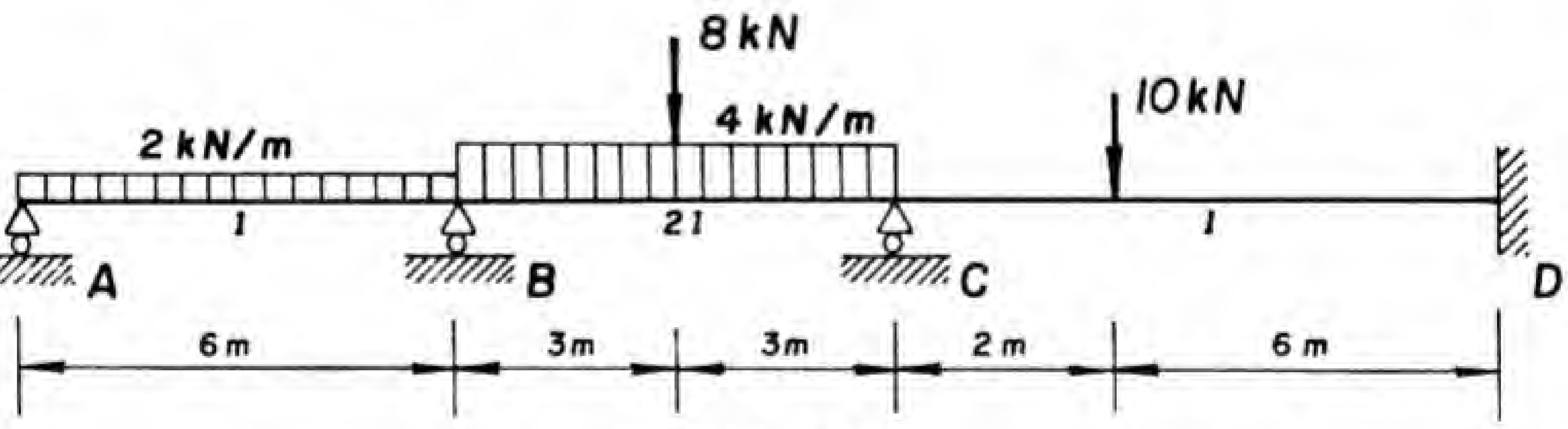


Figure P4.1

4.2 Determine the joint moments of the frame shown in Fig. P3.2.

4.3 Draw the bending moment of the frame shown in Fig. P4.3.

(Ans: $M_A = -14.72$ kN m

$M_B = +10.19$ kN m

$M_C = -6.73$ kN m)

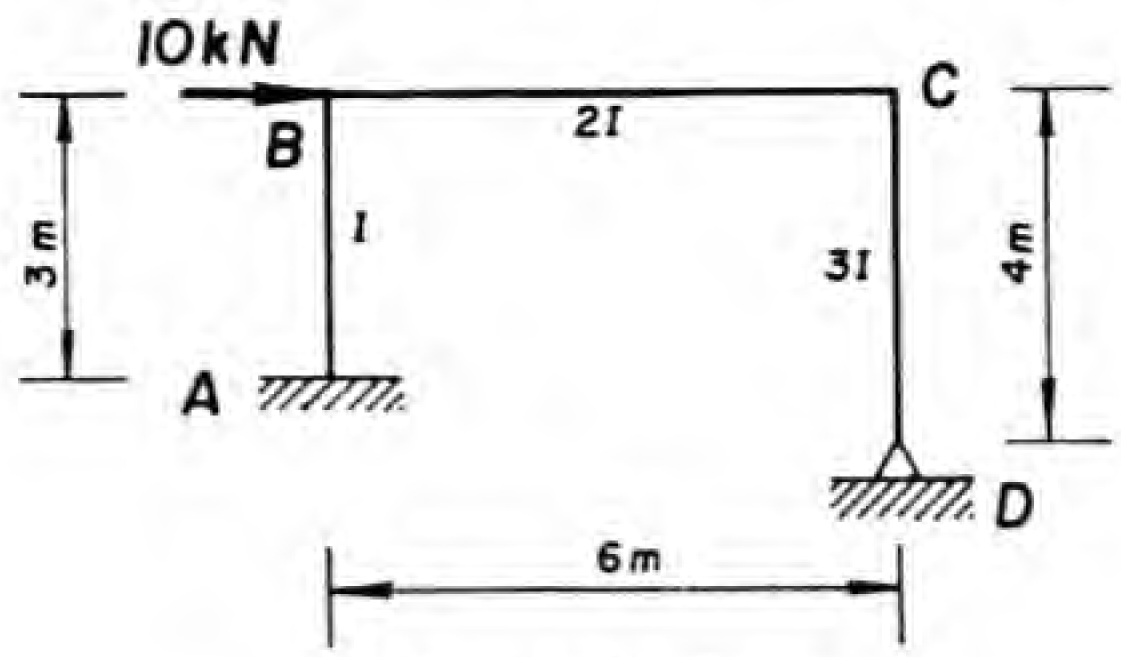


Figure P4.3

4.4 Calculate the support moments of the frame of Fig. P4.4.

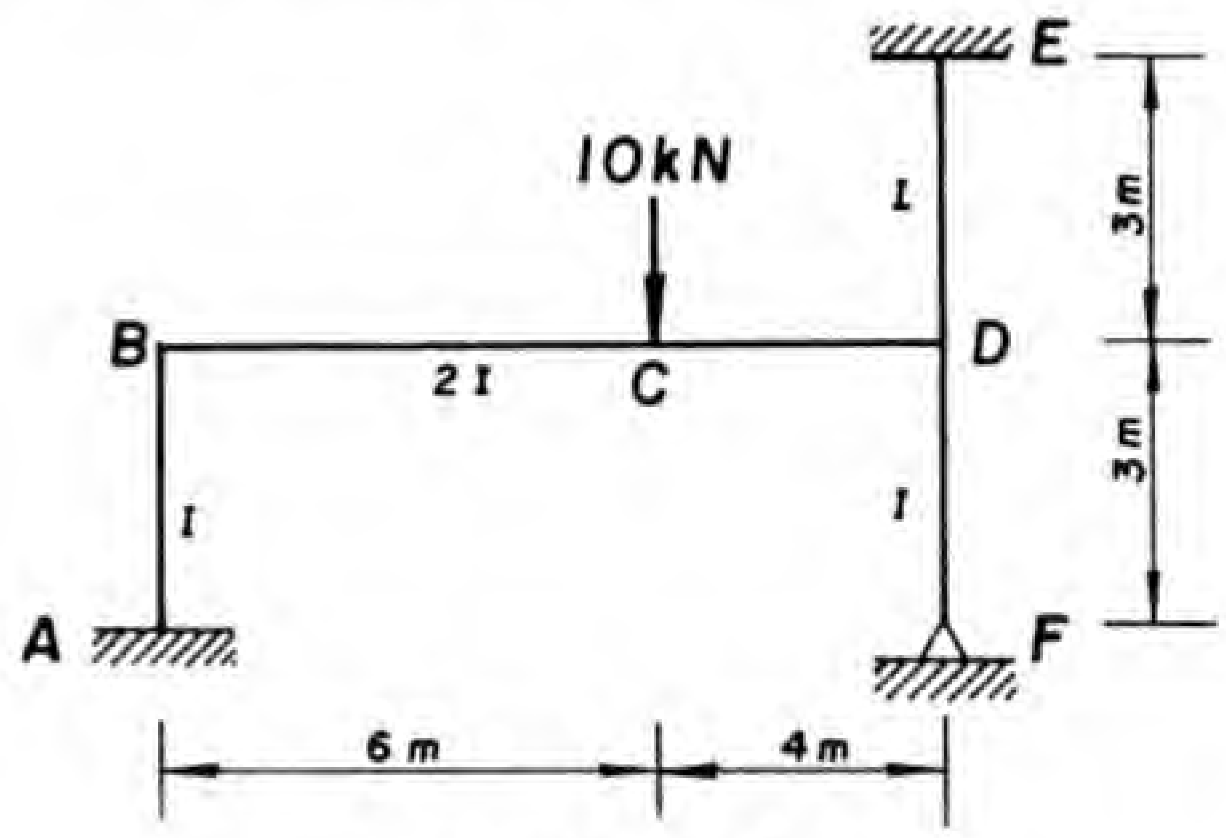


Figure P4.4

4.5 Find the joint moments of the split-frame shown in Fig. P4.5.

(*Ans*: $M_A = -7.93$ kN m

$M_B = +6.08$ kN m

$M_C = -6.65; \quad -1.69; \quad -4.96$ kN m

$M_D = +7.37$ kN m

$M_E = +3.49$ kN m

$M_F = -2.48$ kN m)

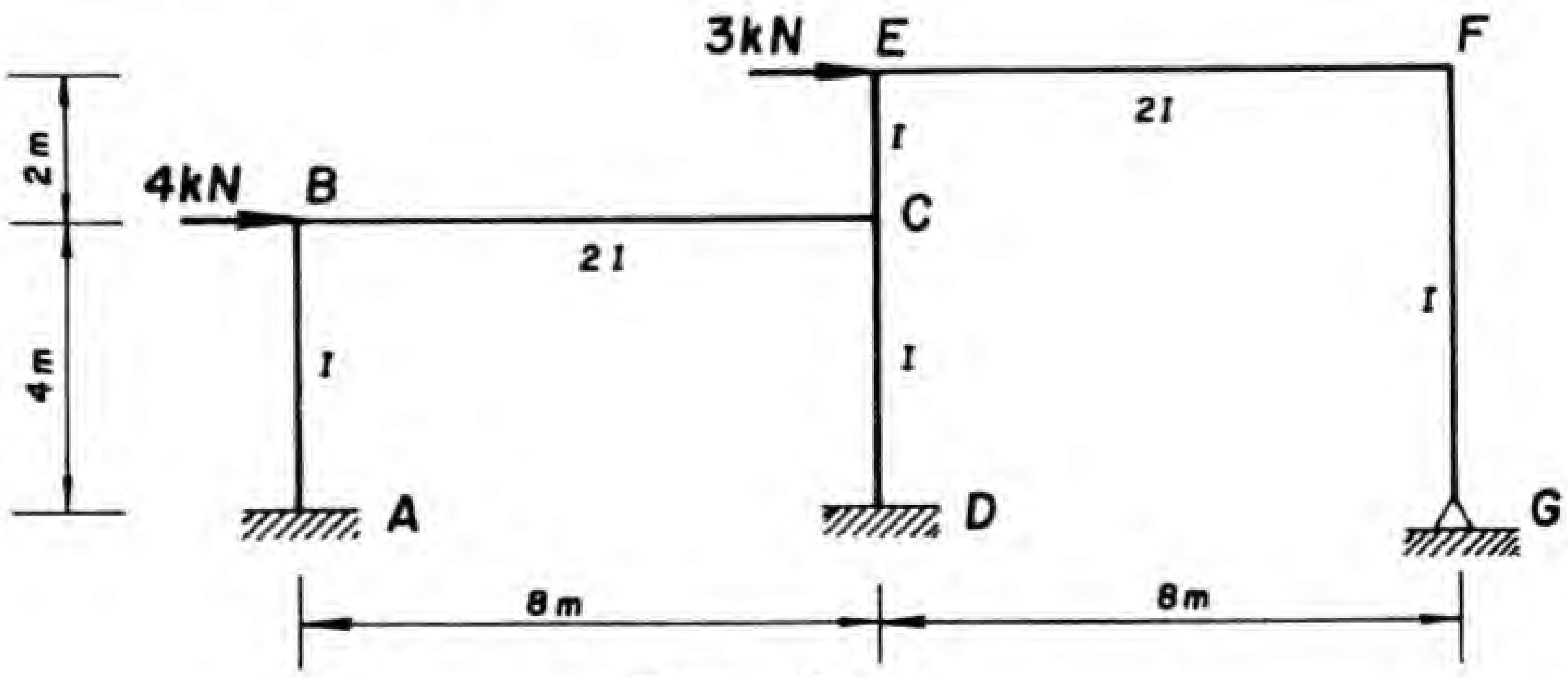


Figure P4.5

4.6 Find the joint moments of the two-storey frame shown in Fig. P4.6 using the cantilever moment distribution method.

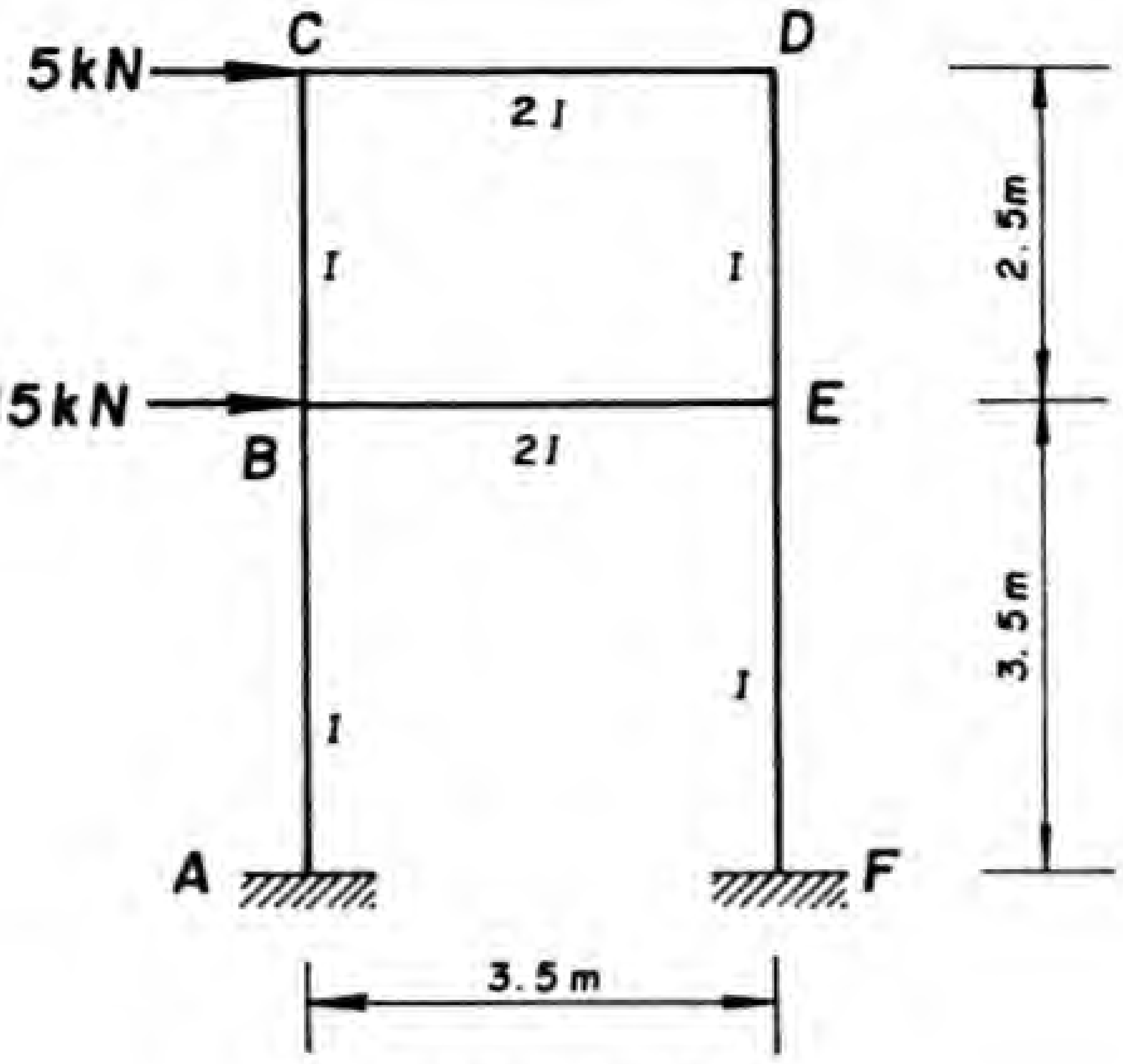


Figure P4.6

4.7 Find the joint moments of the two-storey frame shown in Fig. P4.7 using the cantilever moment distribution method.

(*Ans*: M_{AB} = +0.63 kN m $\quad$ M_D = –5.35 kN m

M_{BA} = +0.19 kN m $\quad$ M_{ED} = +1.69 kN m

M_{BE} = –3.25 kN m $\quad$ M_{EB} = +2.46 kN m

M_{BC} = +3.44 kN m $\quad$ M_{EF} = –0.78 kN m

M_C = –3.6 kN m $\quad$ M_{FE} = –0.34 kN m)

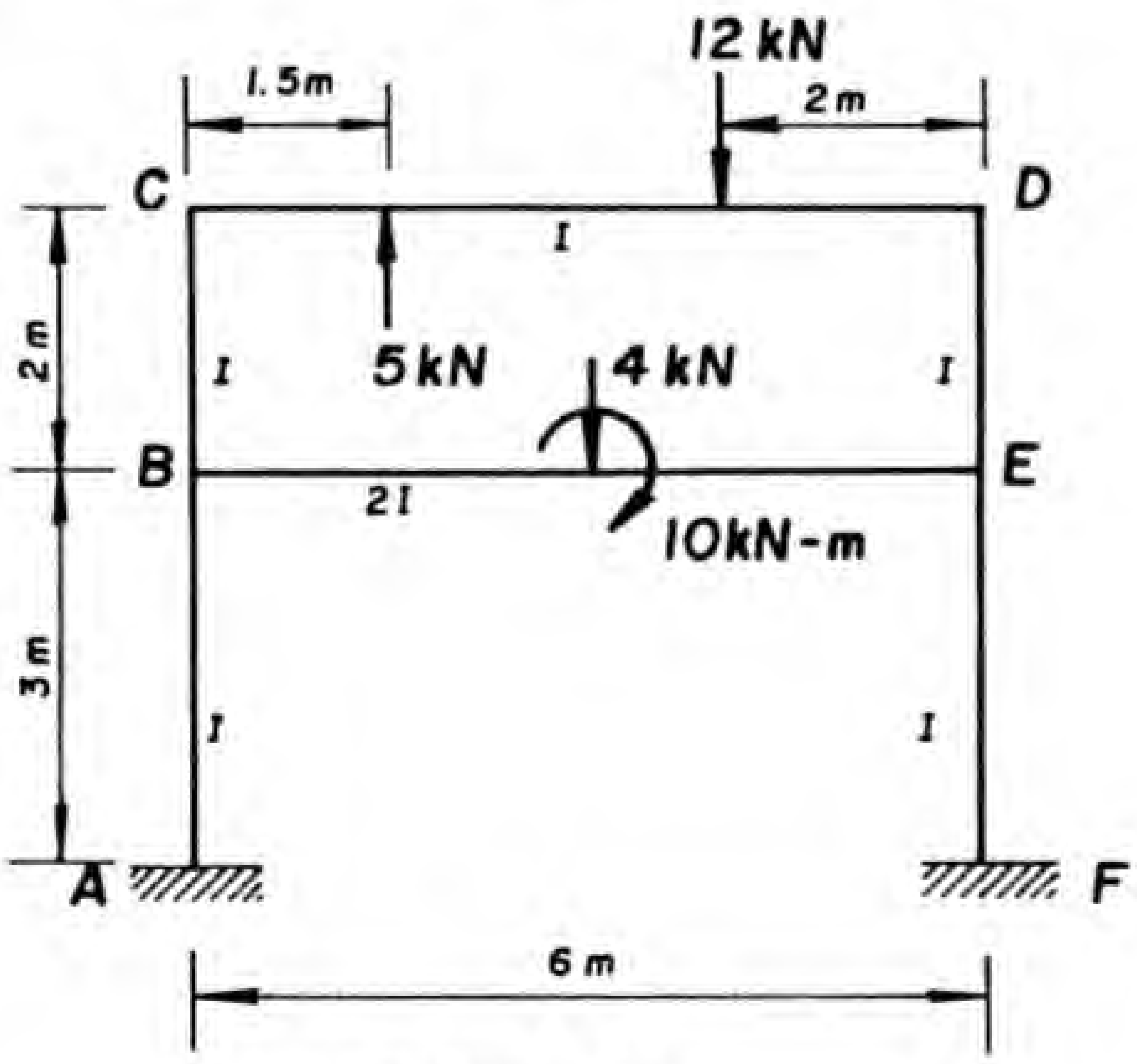


Figure P4.7

5. Kani Method of Moment Distribution

5.1 INTRODUCTION

The moment distribution by the Cross and Kani methods are both iterative procedures to solve the slope deflection equations. However, the Cross method obtains the unknown end moments by iterating the moment increments, while Kani's method iterates the end moments themselves as the unknowns. Kani's method consists of carrying out a single operation applied repeatedly at the joints of a structure in an arbitrary sequence. Using Kani's method the results may also be obtained to any desired accuracy by continuing the calculations a sufficient number of times. In addition to its simplicity, the method has the advantage of acquiring a *built-in* error elimination scheme. Moreover, the method is more suitable for frames with high degree of redundancy, including frames with sidesway, since only one set of computations is necessary. For such frames the required computational effort by Kani method is minimal when compared to other methods.

5.2 FRAMES WITHOUT SIDESWAY

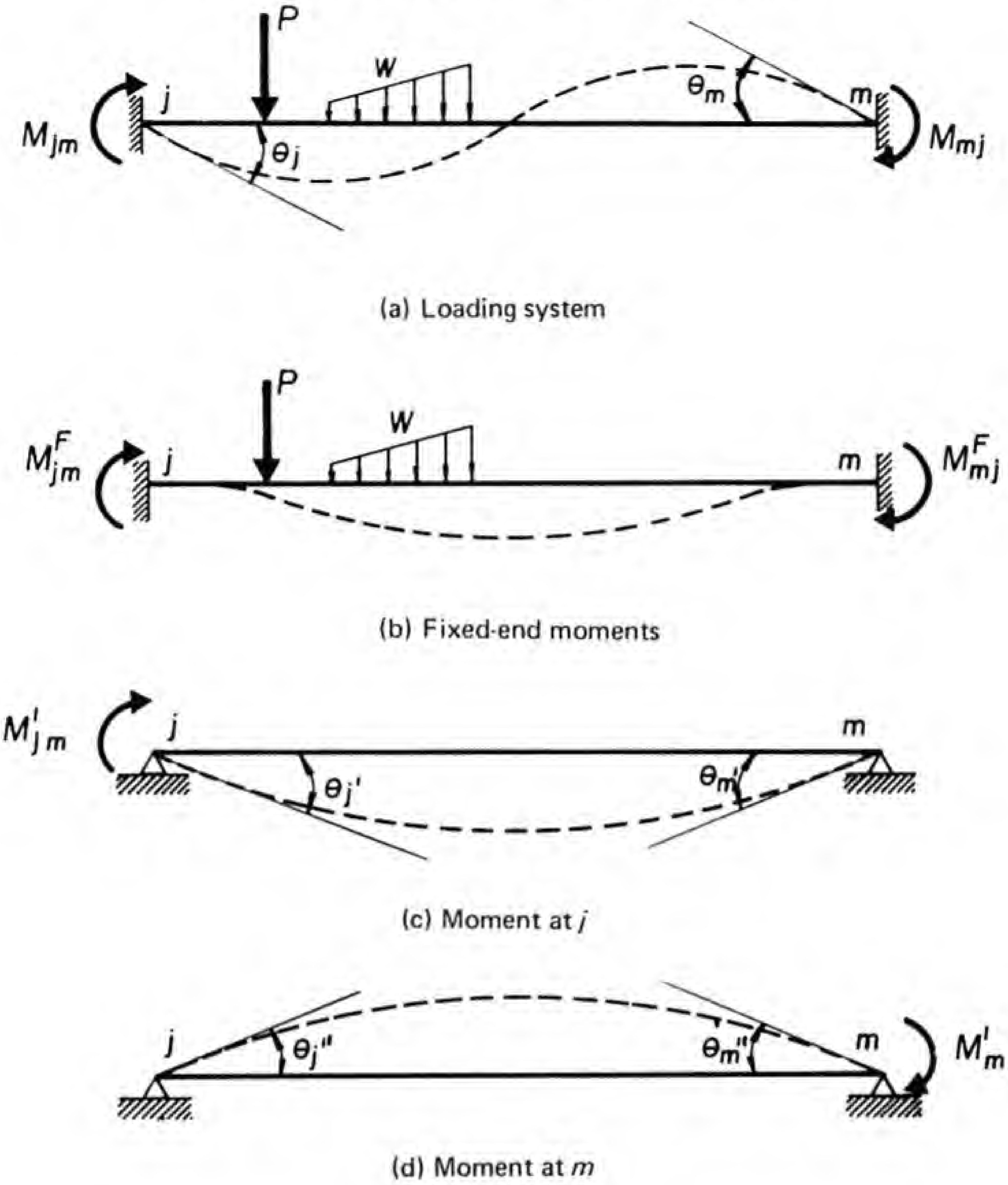
Consider member j–m as integral part of a frame (Fig. 5.1) and that there are many such members meeting at joint j so that m is the general designation, for the far ends. Let M_{jm} and M_{mj} be the end moments due to the applied loads at joints j and m, respectively.

The slope deflection equation for member j–m at joint j which is considered part of a frame without sidesway is written as

$$M_{jm} = M_{jm}^{F} - 2EK_{jm}(2\theta_j + \theta_m) \qquad [5.1]$$

The same equation may be written in the form

$$M_{jm} = M_{jm}^{F} + 2M'_{jm} + M'_{mj} \qquad [5.2]$$



(a) Loading system

(b) Fixed-end moments

(c) Moment at j

(d) Moment at m

Figure 5.1

where

$$M'_{jm} = -2EK_{jm}\theta_j \qquad\qquad M^F_{jm} \tau\Sigma_m$$

$$M'_{mj} = -2EK_{jm}\theta_m$$

Since M'_{jm} constitutes the contribution by θ_j to the total moment M_{jm} it is referred to as the *rotation contribution* of the end m to M_{jm}.

For any joint j where the number of m members are joined to it, the joint being in equilibrium of end-moments gives

$$\sum_m M_{jm} = 0$$

or

$$\sum_m M^F_{jm} + \sum_m (2M'_{jm} + M'_{mj}) = 0 \qquad [5.3]$$

Defining M_j the algebraic sum of the fixed-end moments at joint j as the *restraint moment*

$$M_j = \sum_m M^F_{jm}$$

Then [5.3] may be written as

$$\sum_m M'_{jm} = -\frac{1}{2}(M_j + \sum_m M'_{mj}) \qquad [5.4]$$

Since M_{jm} for any member must be proportional to the relative stiffness of the member, the moment in any member j–m is

$$M'_{jm} = \frac{K_{jm}}{\Sigma K_{jm}} \sum_m M'_{jm}$$

$$= -\frac{1}{2}\left(\frac{K_{jm}}{\Sigma K_{jm}}\right)(M_j + \sum_m M'_{mj}) \qquad [5.5]$$

Denoting the term $-\frac{1}{2}(K_{jm}/\Sigma K_{jm})$ as the *rotation factor* R_{jm}, [5.5] may be written as

$$M'_{jm} = R_{jm}(M_j + \sum_m M'_{mj}) \qquad [5.6]$$

Equation [5.6] forms the basis for Kani's method for frames without sidesway where the rotation contributions are evaluated for member j–m. These contributions and the fixed-end moments are then added algebraically to determine M_{jm} (see [5.1]). But M'_{jm} must initially be evaluated. However, their final values may be determined by successive approximations using the Gauss–Seidel iteration scheme where the rotation contributions proceed from estimated values (starting with initial zero-approximation), the subsequent values being obtained with better approximation. The iteration is terminated when the latest approximation furnishes a value acceptably close to the preceding result.

Kani's method for frames without sidesway may be summarised as follows:

(a) Determine.the fixed-end moments of all members. At each joint evaluate the resultant fixed-end moment, *restraint moment*, ΣM_j.
(b) Calculate the *relative stiffnesses* (K values) of all members using the Gauss–Seidel iteration scheme.
(d) Evaluate the final end-moments using [5.2].

Hinged-End Members

In a structure which contains a hinge located at one end, the stiffness of the member becomes *three-fourths* of the stiffness of a corresponding beam fixed at both ends. Such members, after the fixed-end moments are determined based on the actual member length and introduced into the calculation scheme, are replaced by fictitious members fixed at the hinged ends with the K values which are three-fourths of the actual K values of the members in the original structures. This substitution is justified since the end moment required to produce a unit rotation in the original member is the same as the substitute with three-fourths stiffness. The hinged-end is then left free and its final end-moment is set to zero.

EXAMPLE 5.1 *Find the joint end moments of the symmetric frame shown in Fig. 5.2.*

Fixed-End Moments

$$M^F_{\mathrm{DE}} = M^F_{\mathrm{EF}} = -M^F_{\mathrm{ED}} = -M^F_{\mathrm{FE}} = \frac{3(6)^2}{12} = 9.0 \text{ kN m}$$

$$M^F_{\mathrm{GH}} = M^F_{\mathrm{HI}} = -M^F_{\mathrm{HG}} = -M^F_{\mathrm{IH}} = \frac{2(6)^2}{12} = 6.0 \text{ kN m}$$

The fixed-end moments are recorded at the corresponding member ends in the scheme of calculation (Fig. 5.3).

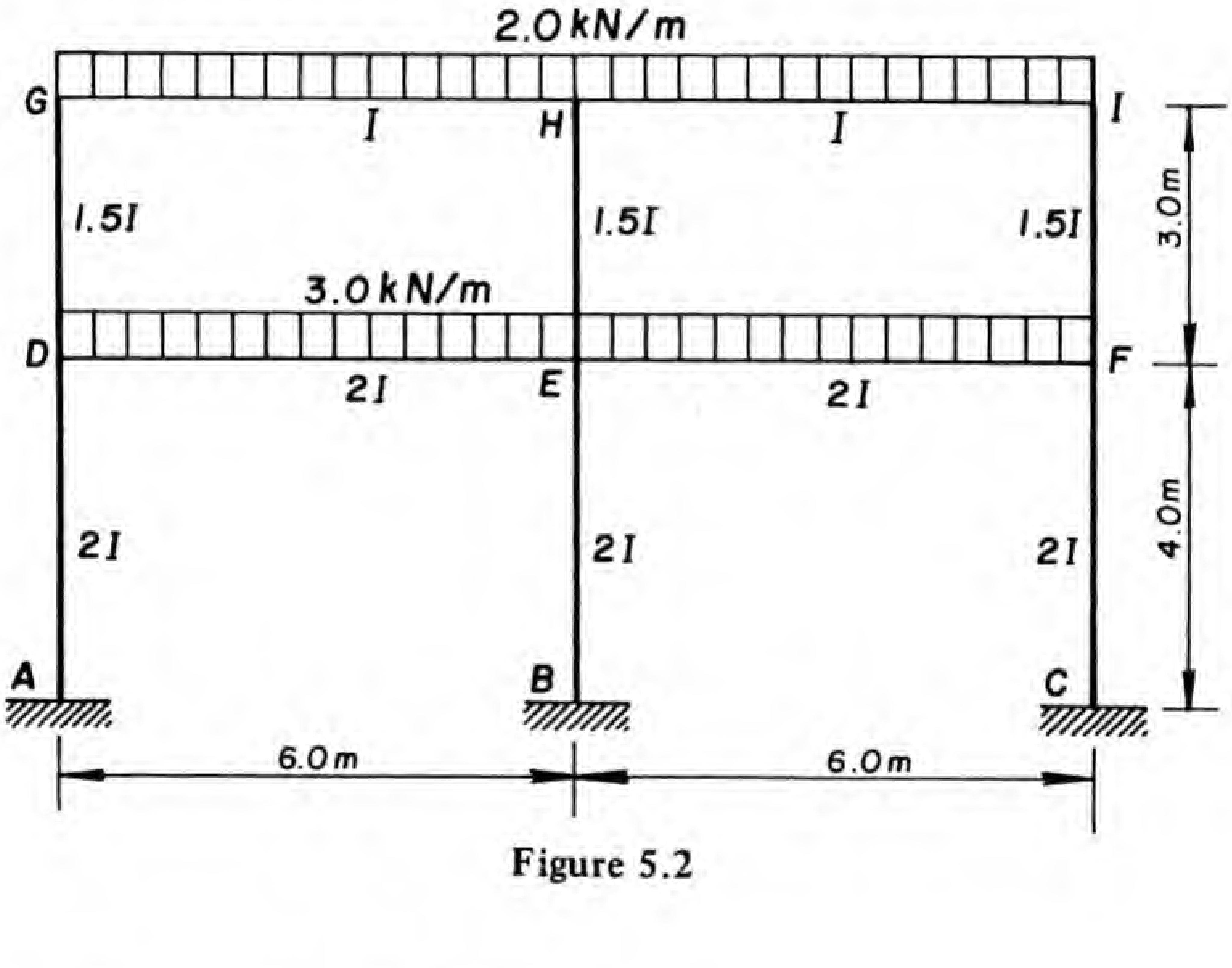


Figure 5.2

Restraint Moments $(M_j = \sum_m M^F_{jm})$

$M_{\mathrm{G}} = +6.0$ kN m

$M_{\mathrm{H}} = (-6.0 + 6.0) = 0.0$

$M_{\mathrm{I}} = -6.0$ kN m

$M_{\mathrm{D}} = +9.0$ kN m

$M_{\mathrm{E}} = 0.0$

$M_{\mathrm{F}} = -9.0$ kN m

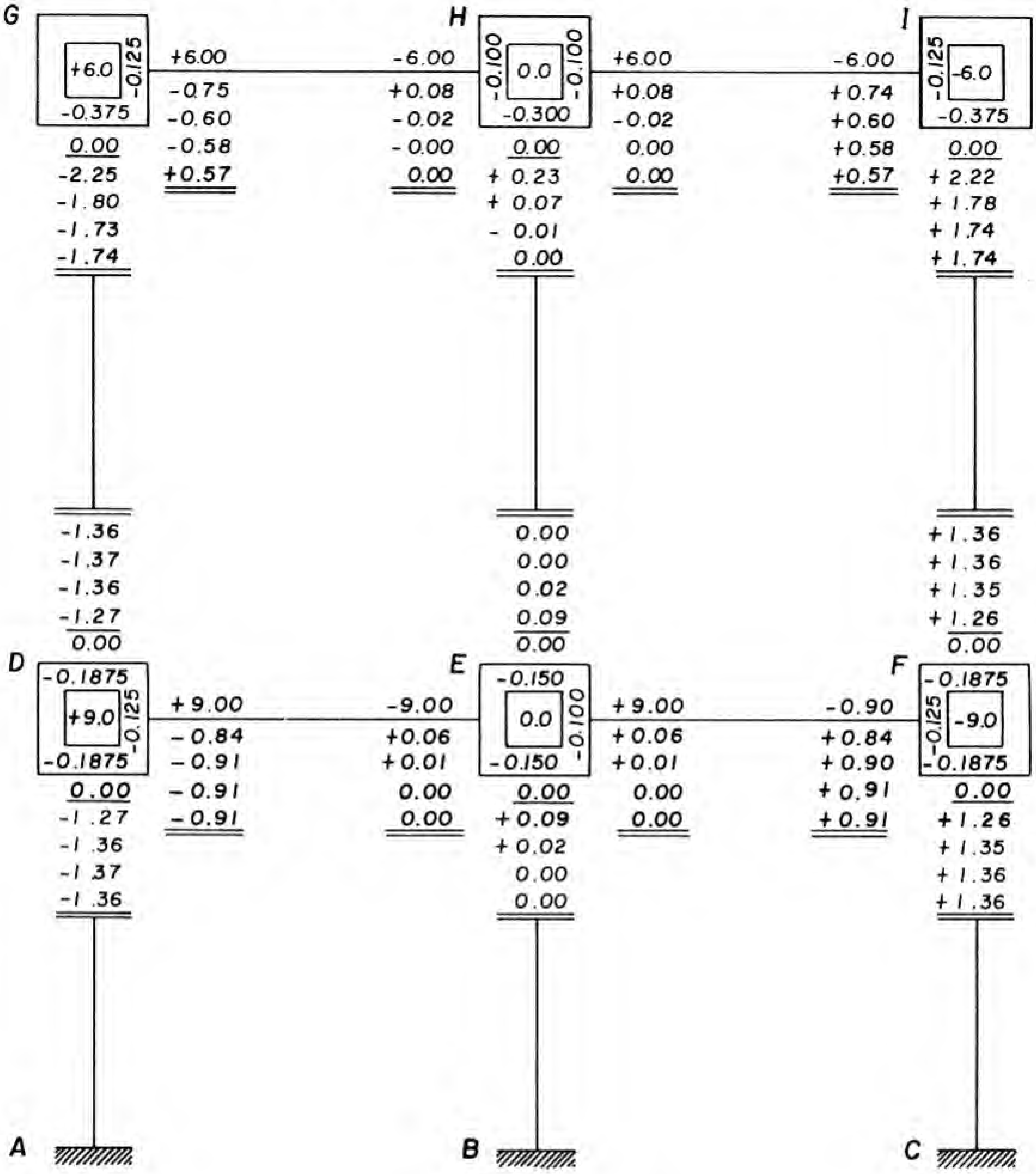


Figure 5.3

Relative Stiffness Values ($K_{jm} = I_{jm}/L_{jm}$)

$$K_{AD} = K_{BE} = K_{CF} = \frac{2}{4} = 0.50$$

$$K_{DG} = K_{EH} = K_{FI} = \frac{1.5}{3} = 0.50$$

$$K_{DE} = K_{EF} = \frac{2}{6} = 0.333$$

$$K_{GH} = K_{HI} = \frac{1}{6} = 0.167$$

Rotation Factors $(R_{jm} = -0.5K_{jm}/\sum_m K_{jm})$

$$R_{\mathrm{DA}} = -\frac{0.50(0.5)}{0.5 + 0.333 + 0.50} = -0.1875$$

$$R_{\mathrm{DG}} = -\frac{0.5(0.5)}{1.333} = -0.1875$$

$$R_{\mathrm{DE}} = -\frac{0.5(0.333)}{1.333} = -0.125$$

Check. $\Sigma R_{jm} = -0.1875 - 0.1875 - 0.125 = -0.50$

Similarly

$R_{\mathrm{GD}} = -0.375$	$R_{\mathrm{GH}} = -0.125$	
$R_{\mathrm{HG}} = -0.10$	$R_{\mathrm{HE}} = -0.30$	$R_{\mathrm{HI}} = -0.10$
$R_{\mathrm{IH}} = -0.125$	$R_{\mathrm{IH}} = -0.375$	
$R_{\mathrm{ED}} = -0.10$	$R_{\mathrm{EH}} = -0.15$	
$R_{\mathrm{EB}} = -0.15$	$R_{\mathrm{EF}} = -0.10$	
$R_{\mathrm{FE}} = -0.125$	$R_{\mathrm{FI}} = -0.1875$	$R_{\mathrm{FC}} = -0.1875$

The rotation factors are recorded at the corresponding ends of the members in the computational scheme (Fig. 5.3).

Rotation Contributions

The contribution to the end moment M_{jm} by the rotation θ_j at joint j is given by

$$M'_{jm} = R_{jm}(M_j + \sum_m M'_{mj})$$

The calculation of the rotation contribution may be started at any joint and continued at other joints in any chosen sequence. The sequence adopted here is GHI–DEF.

First Cycle

(a) Joint G Since the joints H and D are initially locked, the contributions to the end moments at G are zero. Thus, the initial values of these joint moments are set to zero or $M'_{\mathrm{HG}} = M'_{\mathrm{DG}} = 0$ and the rotation contributions at G are

$$M'_{\mathrm{GH}} = -0.125\ (+6.0 + 0.0 + 0.0) = -0.75\ \mathrm{kN\ m}$$

$$M'_{\mathrm{GD}} = -0.375\ (+6.0 + 0.0 + 0.0) = -2.25\ \mathrm{kN\ m}$$

These rotation contributions are entered at the joint G below the respective fixed-end moment as shown in Fig. 5.3.

(b) Joint H At this joint again, $M'_{EH} = M'_{IH} = 0.0$. But, $M'_{GH} = -0.75$, as computed above at joint G. Therefore

$$M'_{HG} = -0.10(0.0 - 0.75 + 0.0 + 0.0) = +0.08 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.3(0.0 - 0.75 + 0.0 + 0.0) = +0.23 \text{ kN m}$$

$$M'_{HI} = -0.1(-0.75) = +0.08 \text{ kN m}$$

In the same manner the calculations are performed at the other joints until the first cycle is completed.

Second Cycle

The results obtained from the first cycle are used to obtain better approximations to the rotation contributions. For example, consider the rotation contribution at joint D:

$$M'_{DG} = -0.187(+9.0 - 1.80 + 0.06) = -1.36 \text{ kN m}$$

$$M'_{DE} = -0.125(9.0 - 1.80 + 0.06) = -0.91 \text{ kN m}$$

$$M'_{DA} = -0.188(9.0 - 1.80 + 0.06) = -1.36 \text{ kN m}$$

A similar calculation is performed at all joints until the second cycle is completed. This procedure is performed until the two successive cycles furnish values differing by only a small acceptable amount. In this particular example, four cycle operations were considered adequate and the complete calculation scheme is shown in Fig. 5.3.

Final End Moments

The final end moments are determined from the relation

$$M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj}$$

Substituting the fixed-end moment values and the rotation contributions obtained from the scheme of calculation:

$$M_{AD} = 0.0 + 2(0.0) - 1.36 = -1.36 \text{ kN m}$$

$$M_{DA} = 0.0 + 2(-1.36) + 0.0 = -2.72 \text{ kN m}$$

$$M_{DE} = +9.0 + 2(-0.91) + 0.0 = +7.18 \text{ kN m}$$

$$M_{DG} = 0.0 + 2(-1.36) - 1.74 = -4.46 \text{ kN m}$$

$$M_{ED} = -9.0 + 2(0.0) - 0.91 = -9.91 \text{ kN m}$$

$$M_{GD} = 0.0 + 2(-1.74) - 1.36 = -4.84 \text{ kN m}$$

$$M_{GH} = +6.0 + 2(-0.58) + 0.0 = +4.84 \text{ kN m}$$

$$M_{HG} = -6 + 2(0.0) - 0.58 = -6.58 \text{ kN m}$$

$$M_{EH} = M_{HE} = M_{EB} = M_{BE} = 0$$

Notice that the final end moments to the right of the structural axis are equal in magnitude but opposite in direction due to the symmetry of both the loading system and the structure itself.

5.3 FRAMES WITH SIDESWAY

When a frame is either structurally unsymmetric or is symmetric with unsymmetrical loading, joint translation or sidesway occurs. Figure 5.4 shows member j–m of a frame with lateral displacement. The rotations at joints j and m are θ_{jm} and θ_{mj} respectively, and Δ_{jm} is the relative lateral displacement between j and m.

The slope deflection equation for the member j–m is

$$\mathrm{M}_{jm} = M_{jm}^F - 2EK_{jm}(2\theta_j + \theta_m - 2\Delta_{jm}/L_{jm}) \qquad [5.7]$$

or

$$M_{jm} = M_{jm}^F + 2M'_{jm} + M'_{mj} + M''_{jm} \qquad [5.8]$$

where

$$M'_{jm} = -2EK_{jm}\theta_j$$

$$M'_{mj} = -2EK_{jm}\theta_m$$

$$M''_{jm} = \frac{6EK_{jm}\Delta_{jm}}{L_{jm}}$$

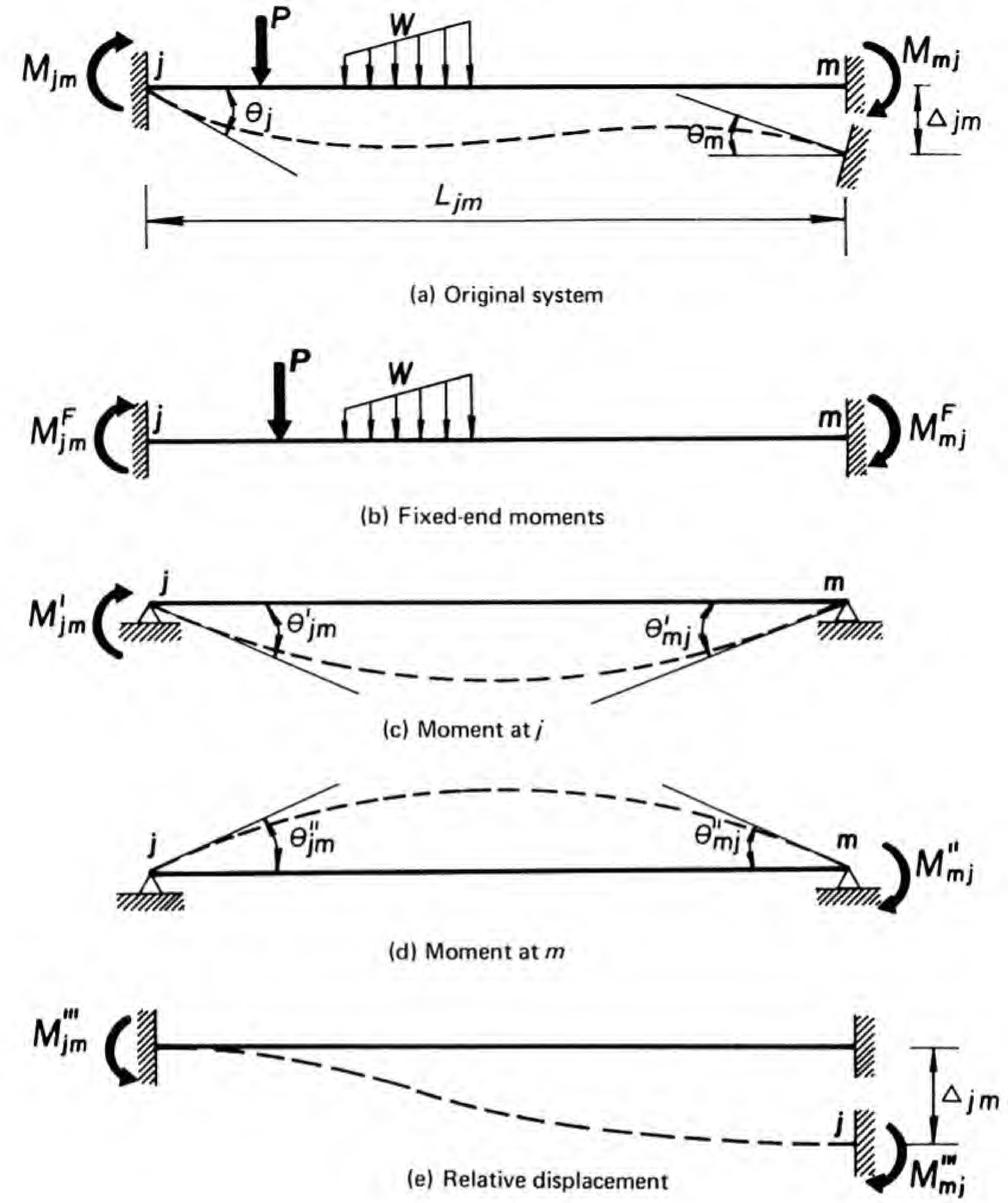
The symbols M'_{jm} and M'_{mj}, respectively, define the *rotation contributions* of the joints j and m to the total moment M_{jm}. In a similar manner, M''_{jm} constitutes the contribution of M_{jm} by the displacement Δ_{jm} and is therefore defined as the *displacement contribution*.

The algebraic sum of the end moments of all members meeting at joint j is zero.

$$\sum_m M_{jm} = 0$$

or

$$\sum_m M_{jm}^F + \sum_m (2M'_{jm} + M'_{mj} + M''_{jm}) = 0 \qquad [5.9]$$



(a) Original system

(b) Fixed-end moments

(c) Moment at j

(d) Moment at m

(e) Relative displacement

Figure 5.4

Defining the algebraic sum of the fixed-end moments at joint j as the *restraint moment*,

$$M_j = \sum_m M^F_{jm}$$

then [5.9] may be written as

$$\sum_m M'_{jm} = -\frac{1}{2} M_j + \sum_m (M'_{mj} + M''_{jm}) \qquad [5.10]$$

Equation [5.10] forms the basis for Kani's method for frames with sidesway where the rotation and displacement contributions must be evaluated for member j–m. These contributions and the fixed-end moments are added

algebraically to determine the final joint moment M_{jm} (see [5.8]). However, while the evaluation of the rotation contributions are easily computed as described in Section 5.2, the determination of the displacement contribution M''_{jm} could be involved depending on the type of loading on the structure and whether there are columns of different length. The cases arising in the determination of displacement contributions are discussed below.

5.3.1 Vertical Loading

Consider the frame shown in Fig. 5.5(a) subjected to vertical loading only. The frame undergoes sidesway due to the unsymmetrically placed vertical loads. The frame analysis may be carried out in two steps as was done in the Cross moment distribution (Section 4.6).

(a) No-sway Solution

As shown in Fig. 5.5(b), artificial joint restraints are applied at storey heights to prevent sidesway. The analysis of the frame without sidesway follows the same procedure presented in Section 5.3. After the joint moments are evaluated the artificial joint restraints may be determined if required, from equilibrium considerations of the shear forces at every storey.

(b) Sway Solution

Since the artificial joint restraints introduced in the *no-sway solution* do not actually exist in the given frame, their presence may be nullified by applying a consistent force system (Fig. 5.5(c)) whose forces are equal in magnitude but opposite in direction to the respective artificial joint restraints.

By cutting horizontally through all columns at the rth storey and from the consideration of equilibrium conditions, the algebraic sum of the column shear must be zero,

$$\sum_r V_{jm} = 0$$

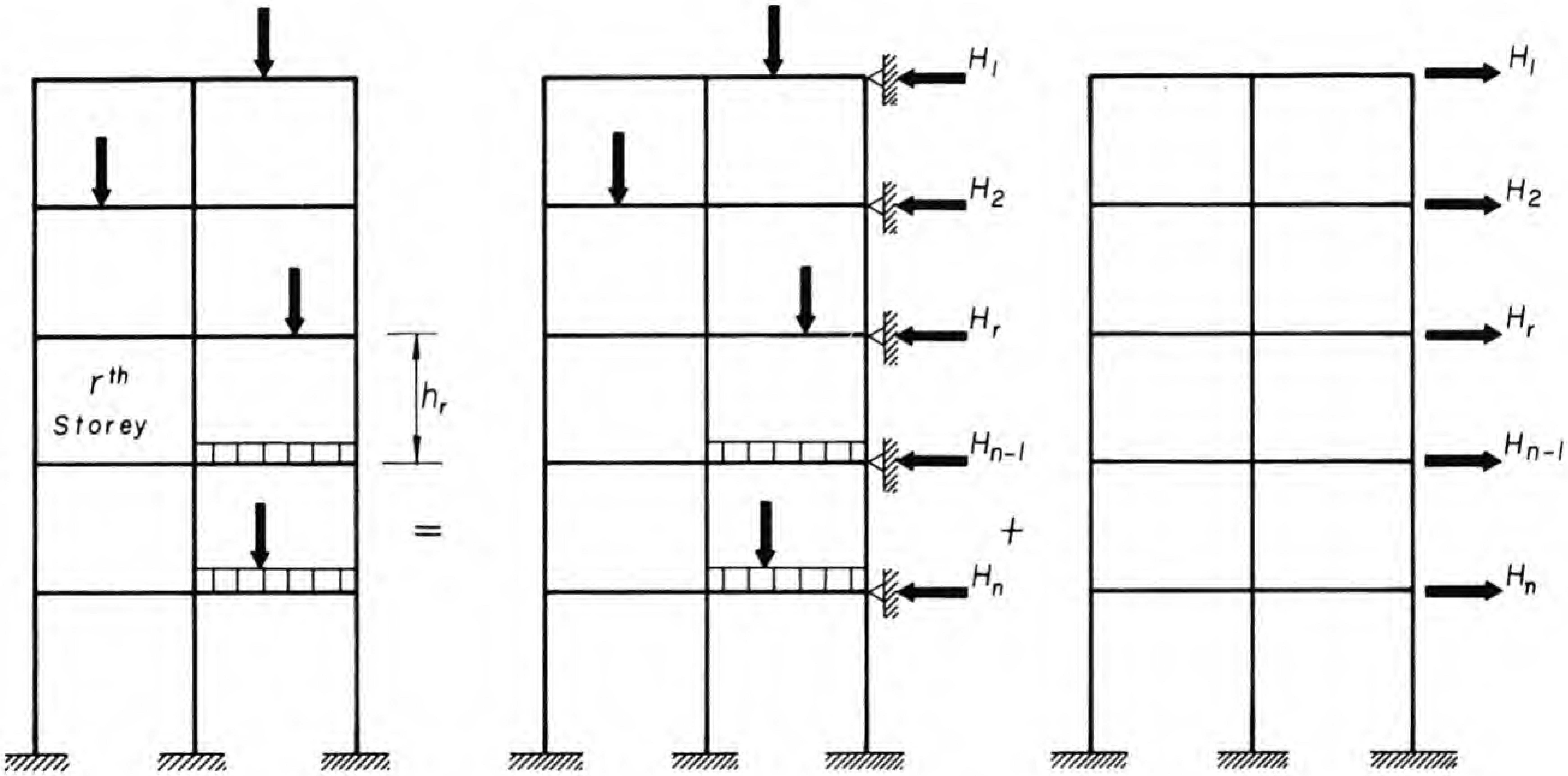
where V_{jm} = shear in column $j–m$ of the rth storey. Let h_r represent the column height of the rth storey, then

$$V_{jm} = \frac{M_{jm} + M_{mj}}{h_r} \qquad [5.11]$$

But

$$M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj} + M''_{jm}$$

$$M_{mj} = M^F_{mj} + 2M'_{mj} + M'_{jm} + M''_{jm}$$



(a) Loaded frame

(b) No – Sway

(c) Consistent lateral force system

Figure 5.5

Since vertical loading only is considered, there are no intermediate loads on the column j–m and the fixed-end moments are zero. Thus

$$M''_{jm} = M''_{mj} = \frac{6EK_{jm}\Delta_{jm}}{L_{jm}} \qquad [5.12]$$

Substituting into [5.11],

$$V_{jm} = \frac{1}{h_r}(3M'_{jm} + 3M'_{mj} + 2M''_{jm}) \qquad [5.13]$$

Taking the sum of the column shears at the rth storey,

$$\sum_r V_{jm} = \sum_r \frac{1}{h_r}[3(M'_{jm} + M'_{mj}) + 2M''_{jm}] = 0 \qquad [5.14]$$

Solving for the sum of the displacement contribution,

$$\sum_r M''_{jm} = -\frac{3}{2}\sum_r (M'_{jm} + M'_{mj})$$

Hence, the algebraic sum of the displacement contribution of all columns at the rth storey is determined to be -1.5 times the sum of the rotation contributions of the column ends of the same storey. Since all the columns at this storey undergo the same lateral displacement, the displacement contribution M''_{jm} for any individual column is proportional to its relative stiffness value (see [5.12]). The displacement contribution of any column is, therefore, obtained by distributing the sum $\sum_r M''_{jm}$ among the columns in the rth storey in proportion to their stiffness values. Thus, for any column j–m

$$M''_{jm} = -1.15\left(\frac{K_{jm}}{\sum_r K_{jm}}\right)\sum_r (M'_{jm} + M'_{mj}) \qquad [5.15]$$

where $\sum_r K_{jm}$ = the sum of the K values of all columns of the rth storey.

In order to make the computation more convenient, and to make an analogy to rotation factor, a *displacement factor* D_{jm} is defined for column j–m as

$$D_{jm} = -1.5\left(\frac{K_{jm}}{\sum_r K_{jm}}\right) \qquad [5.16]$$

Equation [5.15] may be written as

$$M''_{jm} = D_{jm}\sum_r (M'_{jm} + M'_{mj}) \qquad [5.17]$$

Since the displacement contribution M''_{jm} and the rotation distribution M'_{jm} and M'_{mj} are interrelated, [5.17] may be solved by using the same convenient Gauss–Seidel iteration scheme. Once the displacement contributions are known the rotation contributions M'_{jm} are obtained from [5.6] and [5.9] as

$$M'_{jm} = R_{jm}[M_j + \sum_r (M'_{mj} + M''_{jm})] \qquad [5.18]$$

and from these, again, the following approximation of the displacement and rotation contributions are calculated until the results of a desired accuracy are obtained.

The final end moments are then obtained by using [5.8].

EXAMPLE 5.2 *Find the joint moments of the symmetric frame subjected to unsymmetric vertical loading shown in Fig. 5.6.*

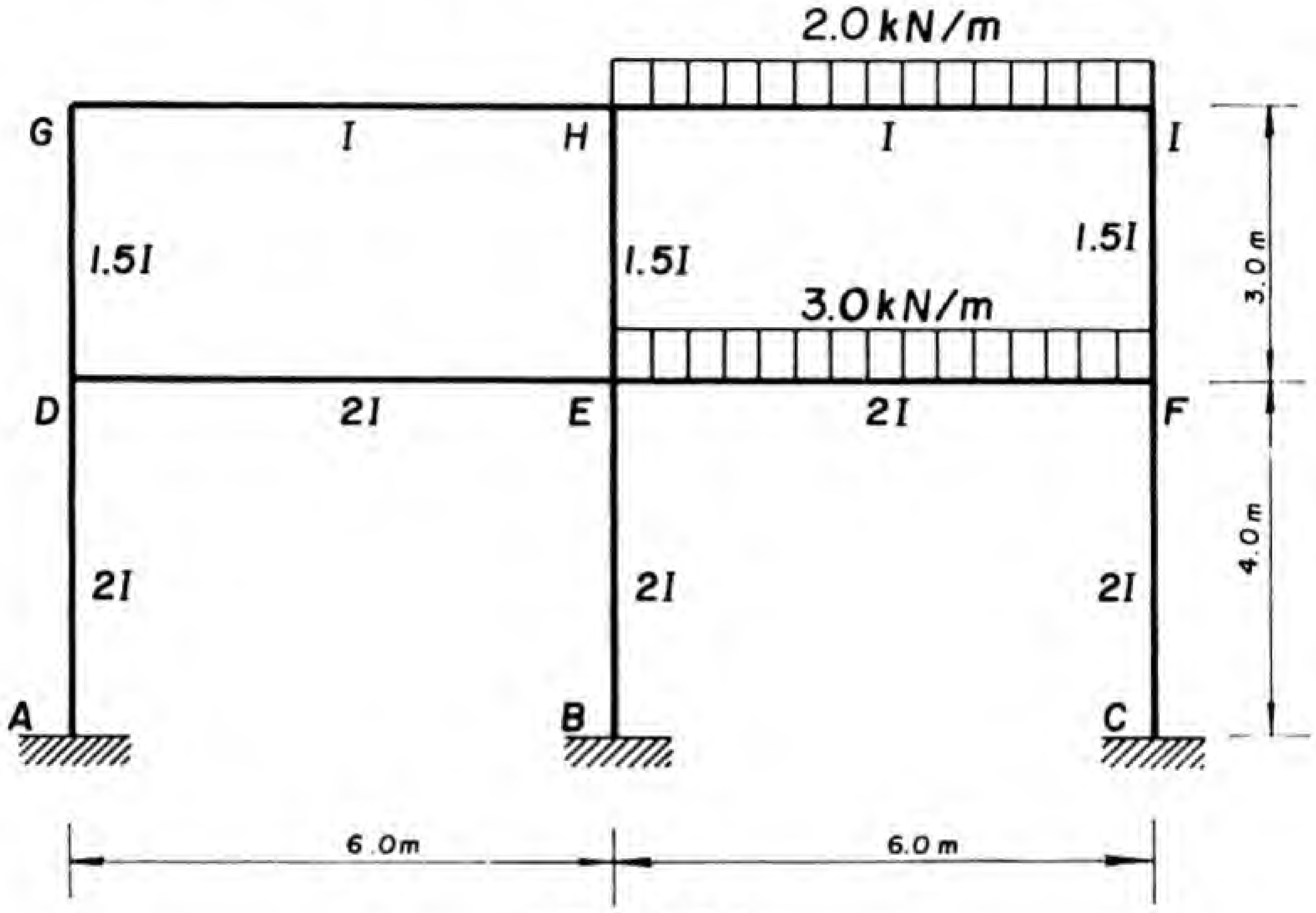


Figure 5.6

Fixed-End Moments

$$M^F_{EF} = -M^F_{FE} = \frac{3(6)^2}{12} = 9.0 \text{ kN m}$$

$$M^F_{HI} = -M^F_{IH} = \frac{2(6)^2}{12} = 6.0 \text{ kN m}$$

Restraint Moments ($M_j = \sum\limits_m M^F_{jm}$)

$$M_G = M_D = 0$$

$$M_H = -M_I = 6.0 \text{ kN m}$$

$$M_E = -M_F = 9.0 \text{ kN m}$$

The relative stiffness values and the rotation factors have the same values as in Example 5.1.

Displacement Factors $(D_{jm} = -1.5K_{jm}/\sum_m K_{jm})$

$$D_{AD} = D_{BE} = D_{CF} = -1.5\left(\frac{0.5}{1.5}\right) = -0.5$$

$$D_{DG} = D_{EH} = D_{FI} = -1.5\left(\frac{0.5}{1.5}\right) = -0.5$$

Notice that the restraint moments M_j and the rotation factors R_{jm} are recorded in the usual manner (Example 5.1) in the computational scheme shown in Fig. 5.7. The displacement factors D_{jm} are also entered at the centre of the relevant columns in the same computational scheme.

First Cycle

The rotation contributions are first computed which are then used to evaluate the displacement contributions.

(a) Rotation Contributions, $M'_{jm} = R_{jm}[M_j + \sum_m (M'_{mj} + M''_{jm})]$

Since M'_{mj} and M''_{jm} are initially known, they are set to zero at the first cycle.

(i) Joint G:

$$\text{Set } M'_{DG} = M'_{HG} = M''_{GD} = 0.0$$
$$M_G = 0.0$$

Thus

$$M'_{GD} = M'_{GH} = -0.375(M_G + M'_{DG} + M'_{HG} + M''_{DG}) = 0.0$$

(ii) Joint H:

$$\text{Set } M'_{EH} = M'_{FH} = M''_{EH} = 0.0$$
$$M'_{GH} = 0.0 \qquad \text{(as found in (i))}$$
$$M_H = +6.00$$

Thus

$$M'_{HG} = -0.100(6.0 + 0.0 + 0.0 + 0.0) = -0.60 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.300(6.0 + 0.0 + 0.0 + 0.0) = -1.80 \text{ kN m}$$

$$M'_{HI} = -0.100(6.0 + 0.0 + 0.0 + 0.0) = -0.60 \text{ kN m}$$

In the same manner all rotation contributions are entered below the relevant fixed-end moments.

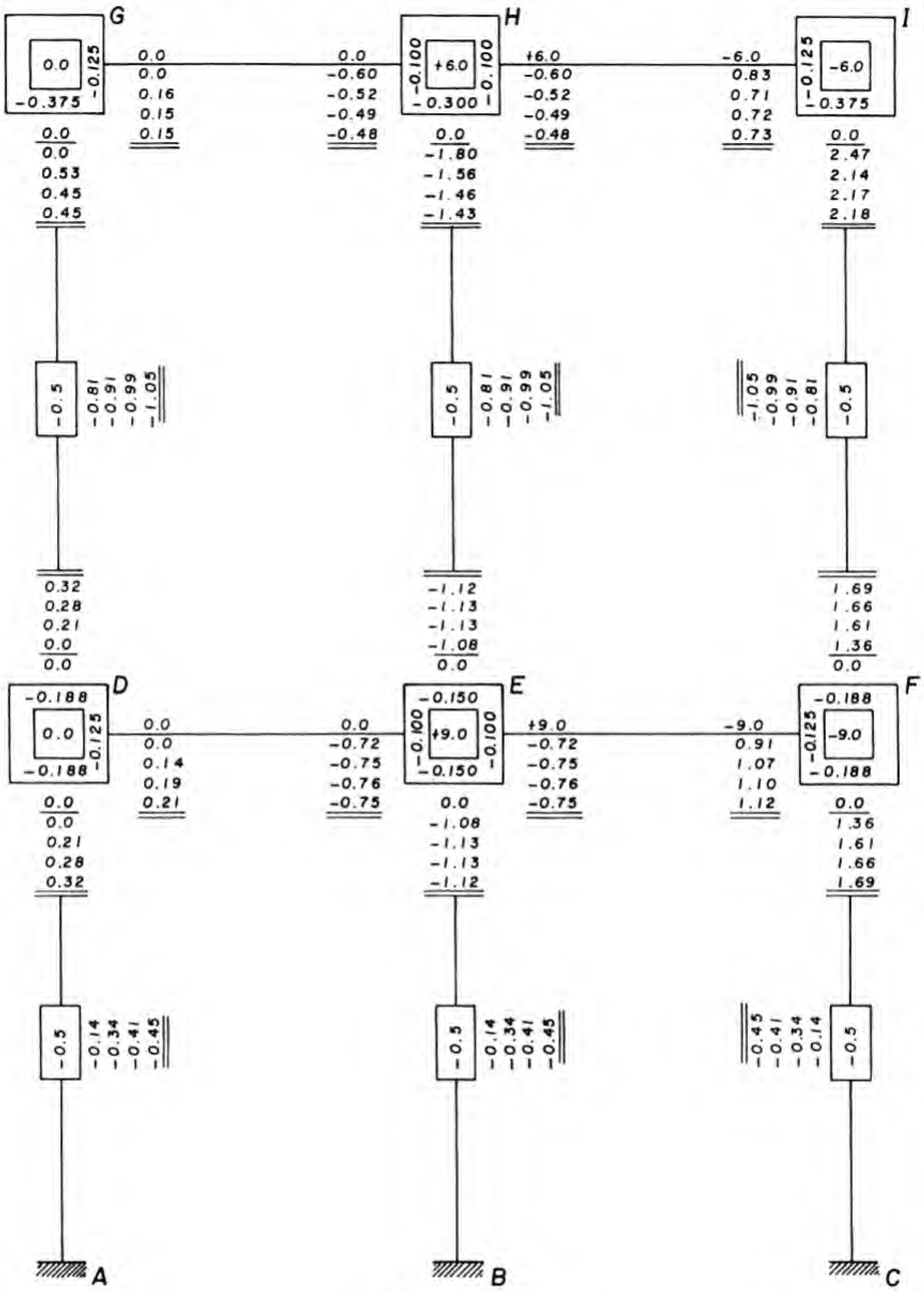


Figure 5.7

(b) Displacement Contributions, $M''_{jm} = D_{jm} \sum_{r} (M'_{jm} + M'_{mj})$

The displacement contributions are computed by multiplying the algebraic term of the rotation contributions of all columns of a storey with the displacement factors of the individual columns.

Thus

$$M''_{DG} = M''_{EH} + M''_{FI} = -0.50(0.0 + 0.0 - 1.80 - 1.08 + 2.48 + 1.36)$$
$$= -0.48 \text{ kN m}$$

Similarly

$$M''_{AD} = M''_{BE} = M''_{CF} = -0.50(0.0 + 0.0 - 1.08 + 0.0 + 1.36 + 0.0)$$
$$= -0.14 \text{ kN m}$$

The displacement contributions are recorded at the middle of the relevant columns as shown in the scheme of calculation (Fig. 5.7).

Second Cycle

The results obtained from the first cycle are used to obtain better approximations to the rotation and displacement contributions.

(a) Rotation Contributions

(i) Joint G:

$$M'_{GD} = -0.375(M_G + M'_{DG} + M'_{HG} + M''_{DG})$$
$$= -0.375(0.00 + 0.00 - 0.60 - 0.48) = 0.41 \text{ kN m}$$

Similarly

$$M'_{GH} = -0.125(0.00 + 0.00 - 0.60 - 0.48) = 0.14 \text{ kN m}$$

(ii) Joint H:

$$M'_{GH} = +0.14 \text{ kN m} \quad \text{(as in (i) above)}$$
$$M''_{EH} = -1.08 \text{ kN m}$$
$$M'_{IH} = +0.83 \text{ kN m}$$
$$M''_{EH} = -0.48 \text{ kN m}$$
$$M_H = +6.0 \text{ kN m}$$

Therefore

$$M'_{HG} = M'_{HI} = -0.100(6.0 + 0.14 - 1.08 + 0.90 - 0.81)$$
$$= -0.54 \text{ kN m}$$
$$M'_{HE} = -0.300(6.0 + 0.18 - 1.08 + 0.83 - 0.48)$$
$$= -1.62 \text{ kN m}$$

Similarly, all rotation contributions of the second cycle are entered below the relevant fixed-end moments.

(b) Displacement Contributions

$$M''_{\mathrm{DG}} = M''_{\mathrm{EH}} = M''_{\mathrm{FI}} = -0.50(0.41 + 0.17 - 1.62 - 1.17 + 2.12 + 1.56)$$
$$= -0.74 \text{ kN m}$$
$$M''_{\mathrm{AD}} = M''_{\mathrm{BE}} = M''_{\mathrm{CF}} = -0.50(0.17 + 0.00 - 1.17 + 0.00 + 1.56 + 0.00)$$
$$= -0.28 \text{ kN m}$$

This completes the second cycle operations. The procedure is then repeated until two successive cycles furnish sets of values differing by a very small acceptable amount. In this particular example, the scheme of computation shows four cycles to be sufficient (Fig. 5.7).

Final End-Moments, $M_{jm} = M^F_{jm} + 2M''_{jm} + M'_{mj} + M''_{jm}$

$$M_{\mathrm{AD}} = 0.0 + 2(0.0) + 0.33 - 0.45 = -0.12 \text{ kN m}$$
$$M_{\mathrm{DA}} = 0.0 + 2(0.33) + 0.0 - 0.45 = 0.21 \text{ kN m}$$
$$M_{\mathrm{DE}} = 0.0 + 2(0.22) + (-0.75) = 0.31 \text{ kN m}$$
$$M_{\mathrm{FE}} = -9.0 + 2(1.12) + (-0.75) = -7.51 \text{ kN m}$$

and so on.

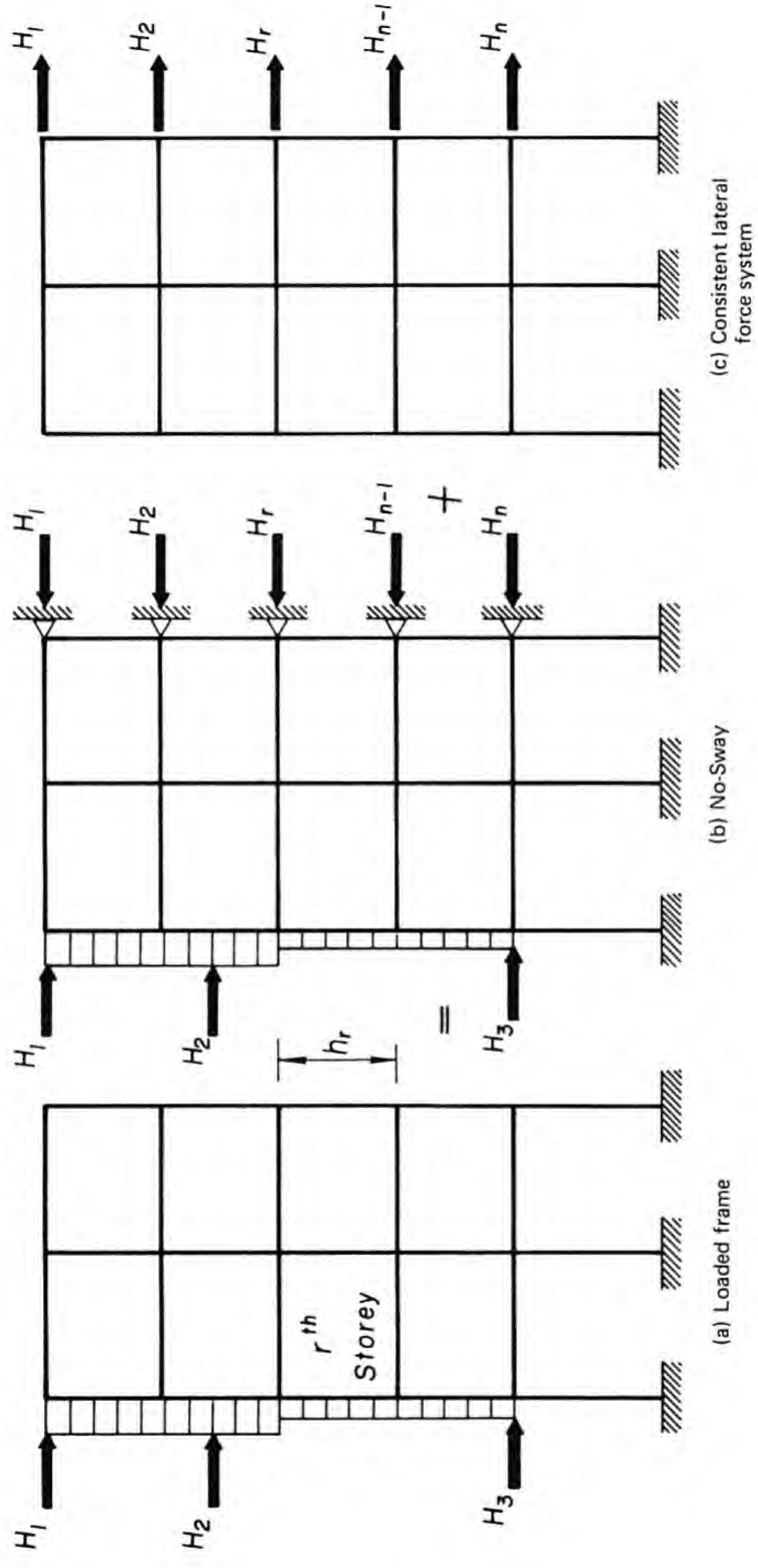
5.3.2 Horizontal Loading

The principle in determining the displacement contributions in frames subjected to horizontal loading remain the same as in vertical loading. However, the presence of horizontal loads on the frame requires an additional effort in the computation.

Consider a frame subjected to horizontal loads applied as shown in Fig. 5.8(a). Again, the analysis may be carried out in two steps:

(a) No-Sway Solution

Artificial joint restraints are applied at storey heights as shown in Fig. 5.8(b) to prevent sidesway. These joint restraints may be determined from no-sway solution (section 5.3).



H1
H2
H3
r th Storey
hr
=
+
Hr
Hn-1
Hn
(a) Loaded frame
(b) No-Sway
(c) Consistent lateral force system

Figure 5.8

(b) Sway Solution

Since the artificial joint restraints do not actually exist, they may be eliminated by applying a consistent force system (Fig. 5.8(c)) whose storey shear at any section is equal in magnitude but opposite in direction to the algebraic sum of the applied horizontal force above that section. Designating the sum of the restraint forces above the rth storey as the storey shear V_r, the horizontal equilibrium condition above the rth storey requires that

$$V_r = H_1 + H_2 + \ldots H_r = \sum_r H$$
$$= \sum_r \frac{1}{h_r}(M_{jm} + M_{mj}) \qquad [5.19]$$

Assuming the columns are not subjected to intermediate horizontal loads (such as by applying equivalent loads at the storey heights that will give the same global effect on the frame), all fixed-end moments become zero. Using [5.8] to expand the end-moments M_{jm} and M_{mj}, [5.19] may be written as

$$V_r = \frac{1}{h_r}\sum_r [3(M'_{jm} + M'_{mj}) + 2M''_{jm}]$$

Rearranging:

$$\sum_r M''_{jm} = -\frac{3}{2}\left[-\frac{V_r h_r}{3} + \sum_r (M'_{jm} + M'_{mj})\right] \qquad [5.20]$$

The quantity $V_r h_r/3$, which is one-third of the product of the storey shear and storey height, is defined as the *storey moment* M_r'.

$$M_r = -\frac{V_r h_r}{3}$$

Equation [5.20] gives the sum of the displacement contributions of all columns in the rth storey. As explained in the case of vertical loads, the moment in any column j–m is obtained by distributing this sum in proportion to their K values. Thus

$$M''_{jm} = D_{jm}[M_r + \sum_r (M'_{jm} + M'_{mj})] \qquad [5.21]$$

Notice that the displacement factors D_{jm} are the same as for vertical loading, so that the displacement contribution in the case of horizontal loading (see [5.21]) differs in the extra term M_r from the case of vertical loading (see [5.17]). Therefore, the analysis of frames subjected to horizontal loading differs from the analysis of frames with vertical loading only by the extra term M_r which must be calculated for each storey and be added algebraically to the sum of the rotation contributions of the two ends of the columns of the storey considered.

EXAMPLE 5.3 *Find the joint moments of the frame subjected to horizontal loads as shown in Fig. 5.9.*

The relative stiffness values, rotation factors, and displacement factors are the same as in Example 5.2 and are recorded in the computational schemes as usual (Fig. 5.9).

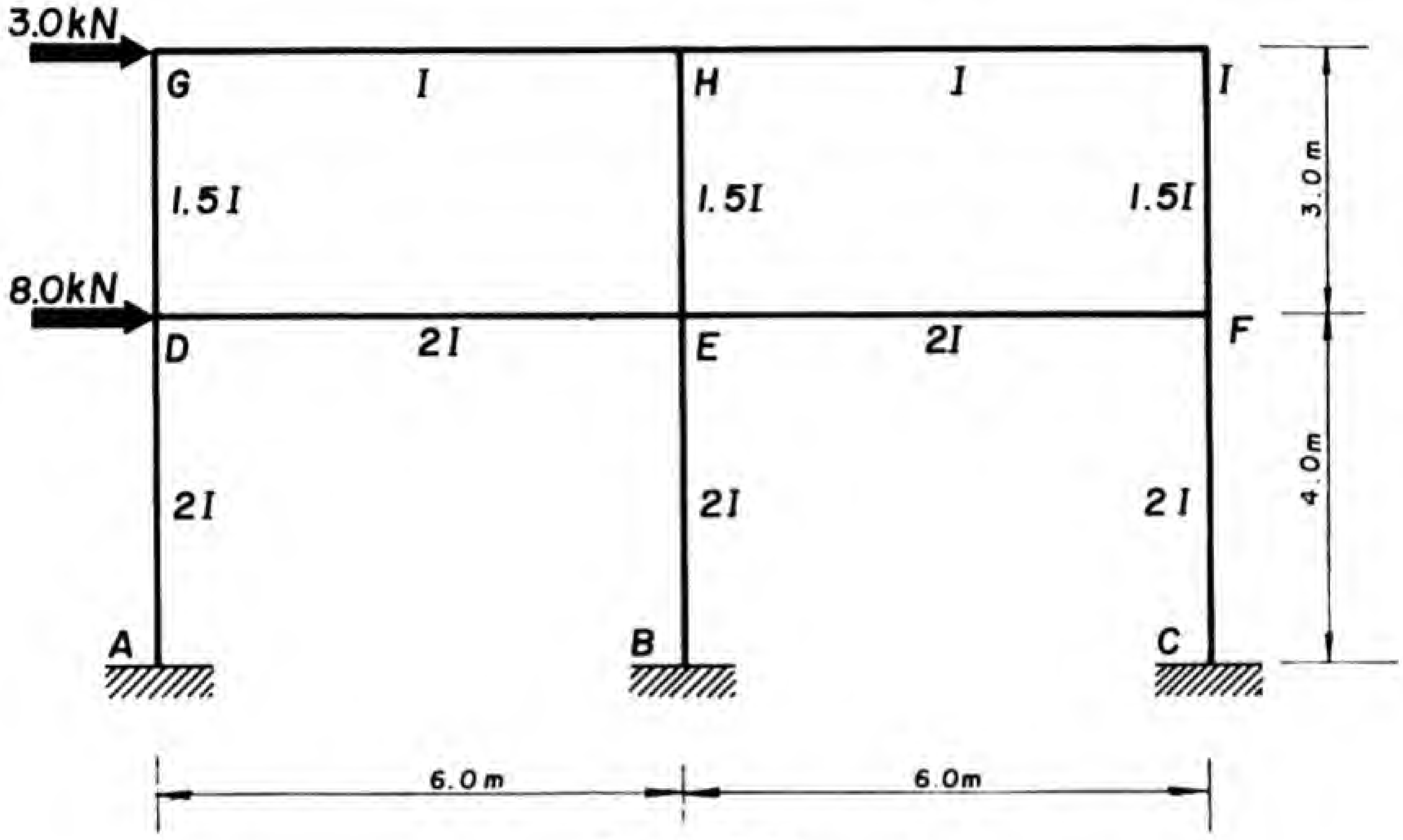


Figure 5.9

Storey Shears and Storey Moments

(i) Second storey

Storey shear, $V_r = 3.0$ kN

$$\text{Storey moment, } M_r = -\frac{V_r h_r}{3} = -\frac{3.0(3.0)}{3} = -3.0 \text{ kN m}$$

(ii) First storey

Storey shear, $V_r = 3.0 + 8.0 = 11.0$ kN

$$\text{Storey moment, } M_r = -\frac{11.0(4.0)}{3} = -14.67 \text{ kN m}$$

The storey moments are recorded in the computational scheme (Fig. 5.10) at the centre of the relevant storey.

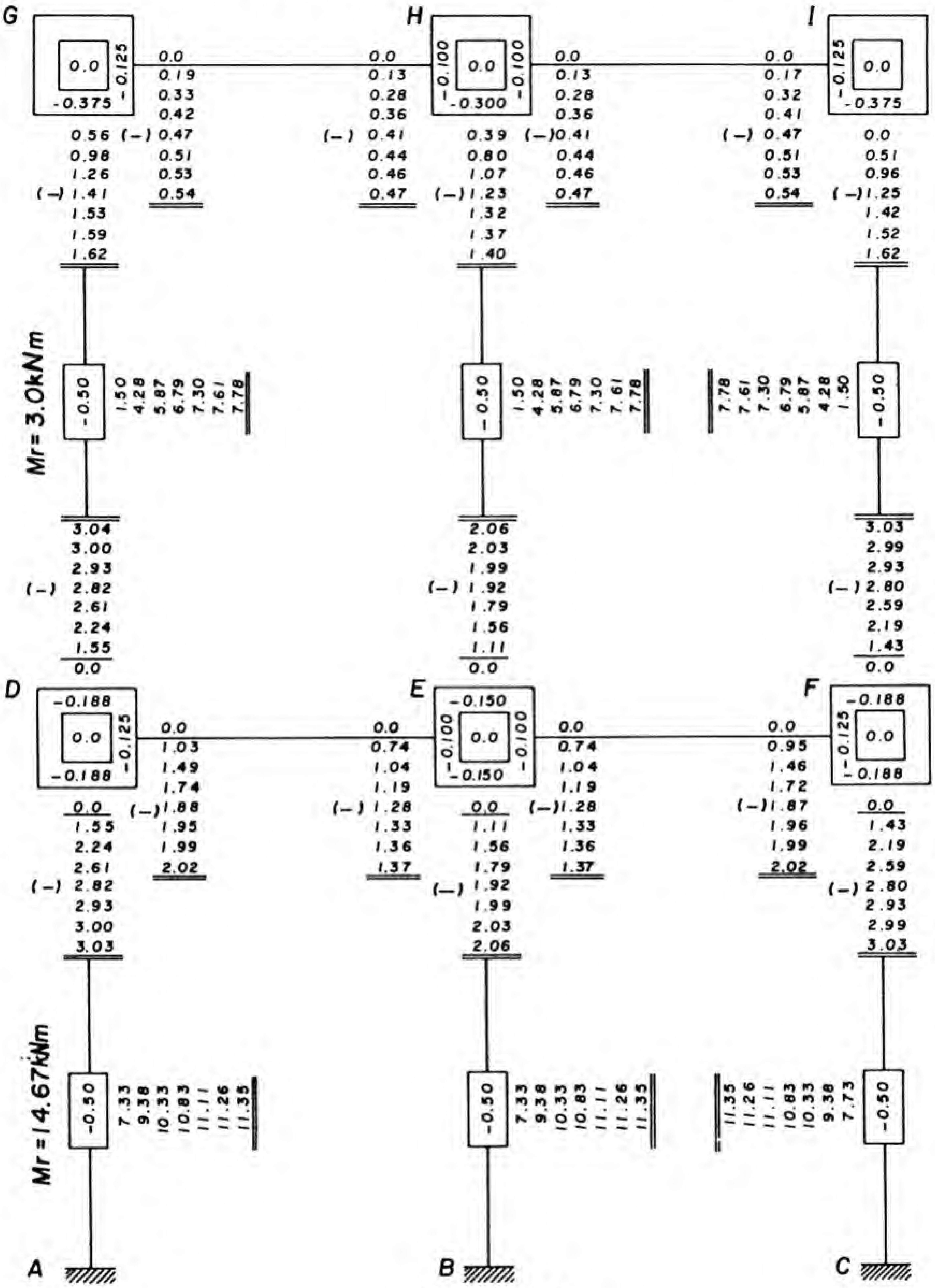


Figure 5.10

Displacement and Rotation Contributions

In frames subjected to horizontal loads, the displacement contributions are usually significantly larger than the rotation contributions. Hence, the displacement contributions are calculated first as illustrated in the following calculations.

First Cycle

(a) Displacement Contribution, $M''_{jm} = D_{jm}[M_r + \sum_r (M'_{jm} + M'_{mj})]$

Since the rotation contributions are initially zero, the displacement contributions are

$$M''_{DG} = M''_{EH} = M''_{FI} = -0.5(-3.0 + 0.0 + 0.0)$$
$$= +1.50 \text{ kN m}$$
$$M''_{AD} = M''_{BE} + M''_{CF} = -0.5(-14.67 + 0.0 + 0.0)$$
$$= +7.33 \text{ kN m}$$

(b) Rotation Contributions, $M'_{jm} = R_{jm}[M'_{mj} + \sum_m (M'_{mj} + M''_{jm})]$

(i) Joint G:

$$\text{At this joint } M_G = M'_{HG} = M'_{DG} = 0$$
$$\text{and } M''_{GD} = -1.50 \text{ kN m}$$
$$M'_{GD} = -0.375(0.0 + 0.0 + 1.50) = -0.56 \text{ kN m}$$
$$M_{GH} = -0.375(0.0 + 0.0 + 1.50) = -0.19 \text{ kN m}$$

(ii) Joint H:

$$\text{At this joint, } M_H = M'_{EH} = M'_{IH} = 0$$
$$M'_{GH} = -0.19 \text{ kN m}$$
$$\text{and } M''_{EH} = +1.50 \text{ kN m}$$

Thus

$$M'_{HG} = -0.100(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -0.13 \text{ kN m}$$

Similarly

$$M'_{HE} = -0.300(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -0.39 \text{ kN m}$$
$$M'_{HI} = -0.100(0.0 - 0.19 + 0.0 + 0.0 + 1.50) = -1.3 \text{ kN m}$$

In the same manner all the rotation contributions are calculated until the first cycle is completed.

Second Cycle

(a) Displacement Contributions

The displacement contributions are obtained by using [5.21] where the results from the first cycle are used to obtain approximations.

$$M''_{DG} = M''_{EH} = M''_{FI} = -0.5(-3.0 - 0.50 - 1.55 - 0.39 - 1.11 - 0.51 - 1.43)$$
$$= +4.28 \text{ kN m}$$

$$M''_{AD} = M''_{BE} = M''_{CF} = -0.50(-14.67 - 1.55 - 1.11 - 1.43)$$
$$= +9.38 \text{ kN m}$$

(b) Rotation Contributions

There are no particular points to be noted here and similar calculations are performed until the second cycle is completed. The procedure is repeated until two successive cycles furnish sets of values differing by a very small acceptable amount. In this particular example, the scheme of computation shows seven cycles to be sufficient (Fig. 5.10).

Final End Moments, $M_{jm} = M^F_{jm} + 2M'_{jm} + M'_{mj} + M''_{jm}$

$$M_{AD} = 0.0 + 2(0.0) - 3.03 - 11.35 = +8.32 \text{ kN m}$$

$$M_{DA} = 0.0 + 2(-3.04) + 0.0 - 11.35 = +5.29 \text{ kN m}$$

$$M_{DE} = 0.0 + 2(-2.02) - 1.37 = -5.41 \text{ kN m}$$

$$M_{DG} = 0.0 + 2(-3.04) - 1.62 + 7.78 = +0.10 \text{ kN m}$$

$$M_{GD} = 0.0 + 2(-1.62) - 3.03 + 7.78 = +1.51 \text{ kN m}$$

$$M_{GH} = 0.0 + 2(-0.54) - 0.47 = -1.55 \text{ kN m}$$

and so on.

5.3.3 Frames With Columns of Unequal Heights

For a frame with a storey containing columns of unequal heights, the calculations of the rotation contributions in all storeys remain the same as described earlier. Also, the computation of the displacement contributions for those storeys with equal heights are not altered. However, in establishing the governing equations

for the calculation of the displacement contribution for the storey with unequal heights, supplementary consideration is needed.

Consider the frame shown in Fig. 5.11. In the storey which has columns of unequal heights, an *arbitrary* column that appears most frequently is taken as the storey height. Let

h_r = *storey height* in the rth storey which has columns of unequal heights

h_{jm} = height of any other column j–m in the rth storey

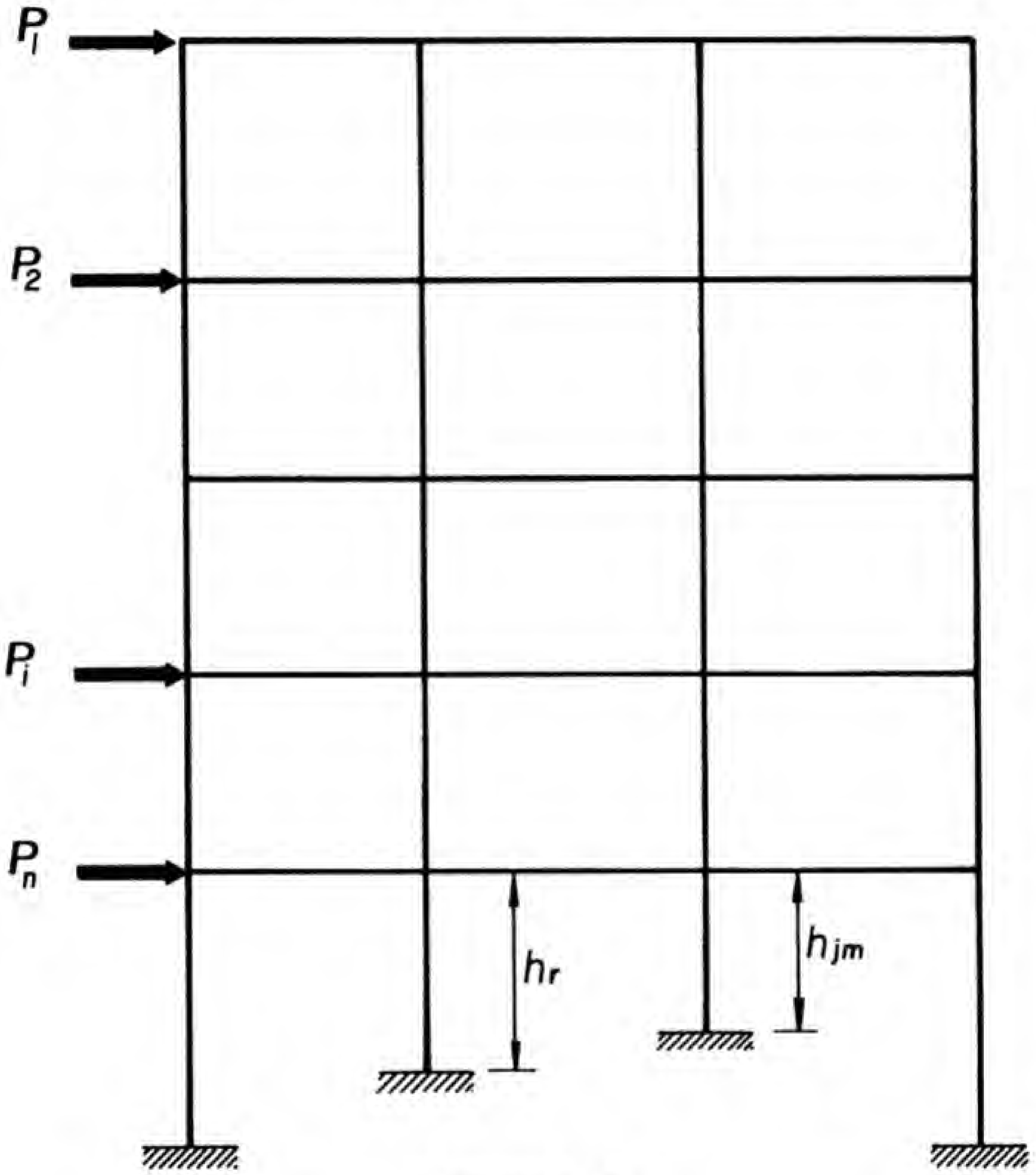


Figure 5.11

Writing the equilibrium condition of the horizontal forces at the rth storey

$$V_r + \sum_r V_{jm} = 0 \qquad [5.22]$$

or

$$V_r + \sum_r \frac{1}{h_{jm}} (M_{jm} + M_{mj})$$

$$V_r h_r + \sum_r \frac{h_r}{h_{jm}} (M_{jm} + M_{mj}) = 0$$

Introducing a factor defined as the *height reduction* factor for the $j-m$ column in the rth storey,

$$C_{jm} = \frac{h_r}{h_{jm}} \qquad [5.23]$$

Equation [5.22] is written as

$$Q_r h_r + \sum_r C_{jm}(M_{jm} + M_{mj}) = 0 \qquad [5.24]$$

Substituting the values of M_{jm} and M_{mj} given by

$$M_{jm} = M_{jm}^F + 2M'_{jm} + M'_{mj} + M''_{jm}$$

$$M_{mj} = M_{mj}^F + 2M'_{mj} + M'_{jm} + M''_{jm}$$

into the shear equations and taking into consideration that the fixed-end moments are zero,

$$V_r h_r + \sum_r C_{jm}(3M'_{jm} + 3M'_{mj} + 2M''_{jm}) = 0$$

Therefore

$$\sum_r C_{jm} M''_{jm} = -1.5(M_r + \sum_r C_{jm}(M'_{jm} + M'_{mj}) \qquad [5.25]$$

where

$$M_r = \frac{V_r h_r}{3} = \text{storey moment}$$

Since

$$M''_{jm} = \frac{6EK_{jm}\Delta_{jm}}{h_{jm}}$$

them M''_{jm} is proportional to K_{jm}/h_{jm} and also to $C_{jm}K_{jm}$. Also, since Δ_{jm} is the same for all columns of the storey under consideration,

$$\frac{M''_{jm}}{\sum_r C_{jm} M_{jm}} = \frac{C_{jm}K_{jm}}{\sum_r C_{jm}^2 K_{jm}} \qquad [5.26]$$

From [5.25] and [5.26], the basic equation for determining the displacement contribution M''_{jm} may be written as

$$M''_{jm} = D'_{jm}\,[M_r + \sum_r C_{jm}(M'_{jm} + M'_{mj})] \qquad [5.27]$$

where

$$D'_{jm} = -1.5\left(\frac{C_{jm}K_{jm}}{\sum_r C_{jm}^2 K_{jm}}\right) = \textit{displacement factor}$$

For the storey with unequal column heights, the following changes must be noted:

(a) a *reduction factor* C_{jm} must be introduced;
(b) a *modified displacement factor* D'_{jm} must be used.

Having introduced these factors, [5.26] and [5.27] along with [5.8] and [5.18] are used to determine the end moments.

5.4 PROBLEMS

5.1 Solve problem 3.1 using Kani's method of moment distribution.

5.2 Solve problem 3.2 using Kani's method of moment distribution.

5.3 Solve problem 3.3 using Kani's method of moment distribution.

5.4 Solve the problem of Example 4.7 using Kani's method of moment distribution.

5.5 Solve the problem of Example 4.8 using Kani's method of moment distribution.

6. Influence Lines for Indeterminate Structures

6.1 INTRODUCTION

The determination of the maximum and sometimes the minimum structural effects of the appropriate load system is an important preliminary step in the analysis and design of structures. Structures subjected to moving or movable loads invariably involve the calculation of the maximum or minimum values of the bending moment, shear force, deflection and so forth, by preparing the influence lines for the various structural effects.

The influence lines for statically determinate structures may be drawn by connecting a few key ordinates with straight lines. However, influence lines for indeterminate structures are curved and therefore cannot be drawn so easily. The first step in preparing the influence lines for the various functional values is to determine the influence lines for the redundants. The next step then is to find the influence lines for any other reaction, moment or shear, etc. that can be computed by statics. The influence lines for different functions in statically indeterminate structures may be obtained using the Müller-Breslau principle backed by computational techniques such as the conjugate beam principle, Cross moment distribution, and energy methods.

6.2 STRUCTURES WITH A SINGLE REDUNDANT REACTION

The influence lines for indeterminate structures may be constructed by using either *statical* or *kinematic* methods. When using the statical method consideration of equilibrium alone is utilised. This may be demonstrated by considering a propped cantilevered beam of uniform cross-section. It is desired to prepare an influence line for the vertical reaction at support B shown in Fig. 6.1.

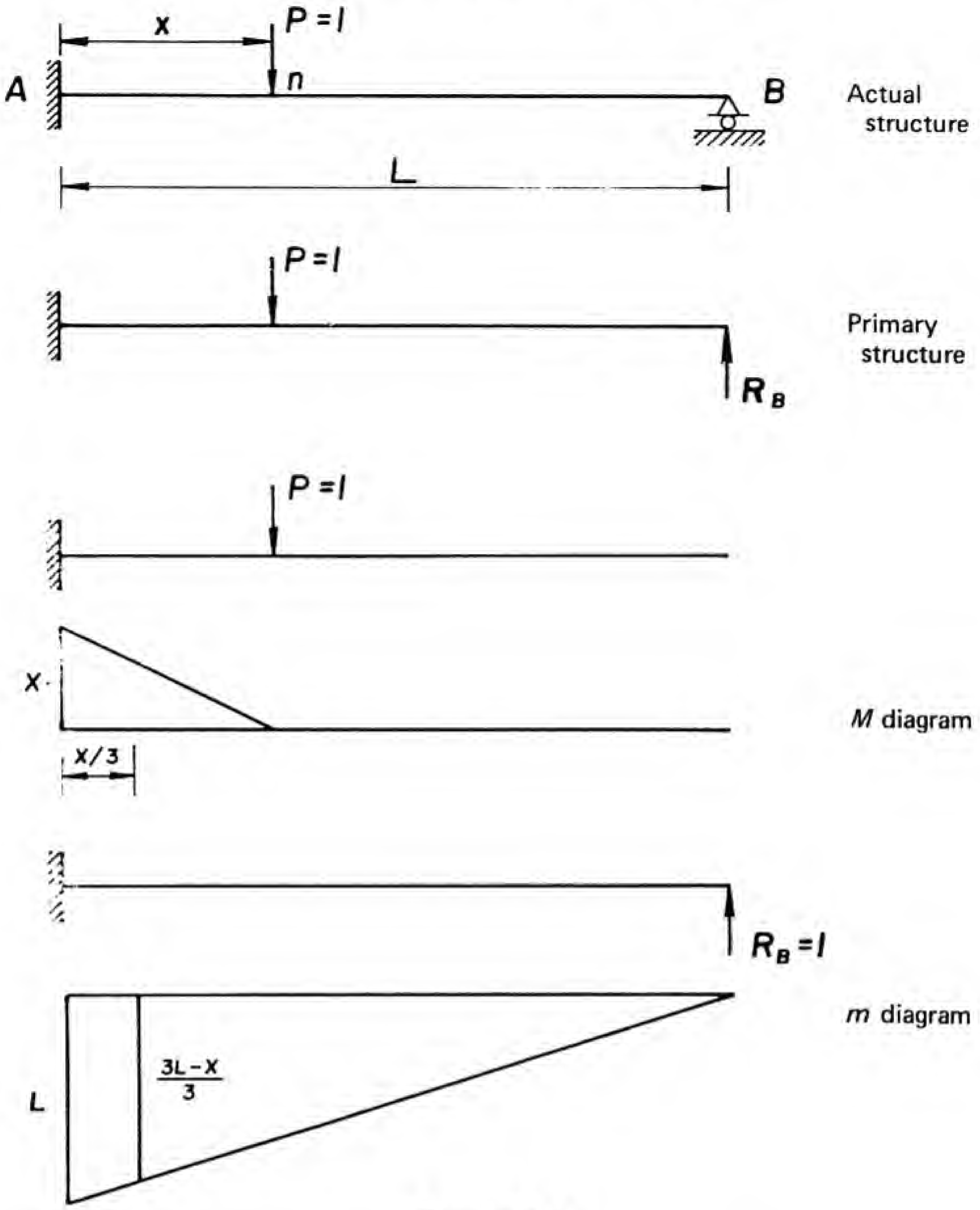


Figure 6.1

The reaction R_B is determined from the compatibility condition

$$R_B\delta_{bb} + \delta_{bo} = 0 \qquad [6.1]$$

from which

$$R_B = -\frac{\delta_{bo}}{\delta_{bb}} \qquad [6.2]$$

The deflections δ_{bo} and δ_{bb} are determined from the M and m diagrams (Fig. 6.1).

$$\delta_{bo} = \int \frac{Mmdx}{EI}$$

$$= -\frac{1}{EI}\left(\frac{x^2}{2}\right)\left(\frac{3L-x}{3}\right) = -\frac{x^2(3L-x)}{6EI}$$

and

$$\delta_{bb} = \int \frac{m^2 dx}{EI}$$

$$= \frac{1}{EI}\left(\frac{L^2}{2}\right)\left(\frac{2L}{3}\right) = \frac{L^3}{3EI}$$

The reaction at B is therefore

$$R_B = -\frac{\delta_{bo}}{\delta_{bb}} = \frac{x^2(3L - x)}{2L^3}$$

This equation gives the value of the reaction R_B when the unit load $P = 1$ is applied at any position along the beam which is therefore the influence line for the reaction at B. Using statics, the influence line for any other reaction, shear or moment may be determined.

Suppose it is required to find the influence line for the shear at the midspan of the beam. From statics, the following may be determined:

$$(V_C)_{\text{left}} = -R_B = -\frac{x^2(3L - x)}{2L^3} \text{ for } 0 < x < L/2$$

$$(V_C)_{\text{right}} = 1 - R_B = 1 - \frac{x^2(3L - x)}{2L^3} \text{ for } L/2 < x < L$$

The influence line for the bending moment at the midspan of the beam may also be evaluated in a similar manner.

The bending moment at midspan is

$$M_C = R_B \frac{L}{2}$$

$$= \frac{x^2(3L - x)}{4L^2} \text{ for } 0 < x < L/2$$

$$M_C = R_B \frac{L}{2} - 1\left(x - \frac{L}{2}\right)$$

$$= \frac{x^2(3L - x) - 4L^2x + 2L^3}{4L^2} \text{ for } L/2 < x < L$$

The ordinates of the influence line at quarter points for the reaction at B, R_B, the shear at midspan, V_C, and the bending moment at midspan, M_C, are given in Table 6.1 and also shown in Fig. 6.2.

To demonstrate the *kinematic* method of constructing influence lines consider the same propped cantilever beam of Fig. 6.1 where the reaction R_B is taken as redundant. Compatibility condition gives

$$R_B\delta_{bb} + \delta_{bo} = 0 \qquad [6.3]$$

Table 6.1 Ordinates of influence lines

x	R_B	V_C	M_C
0	0	0	0
$0.25L$	0.0859	-0.0859	$0.0430L$
$0.50L$	0.3125	-0.3125 or $+0.6875$	$0.1563L$
$0.75L$	0.6328	$+0.2975$	$0.0664L$
L	1.0	0	0

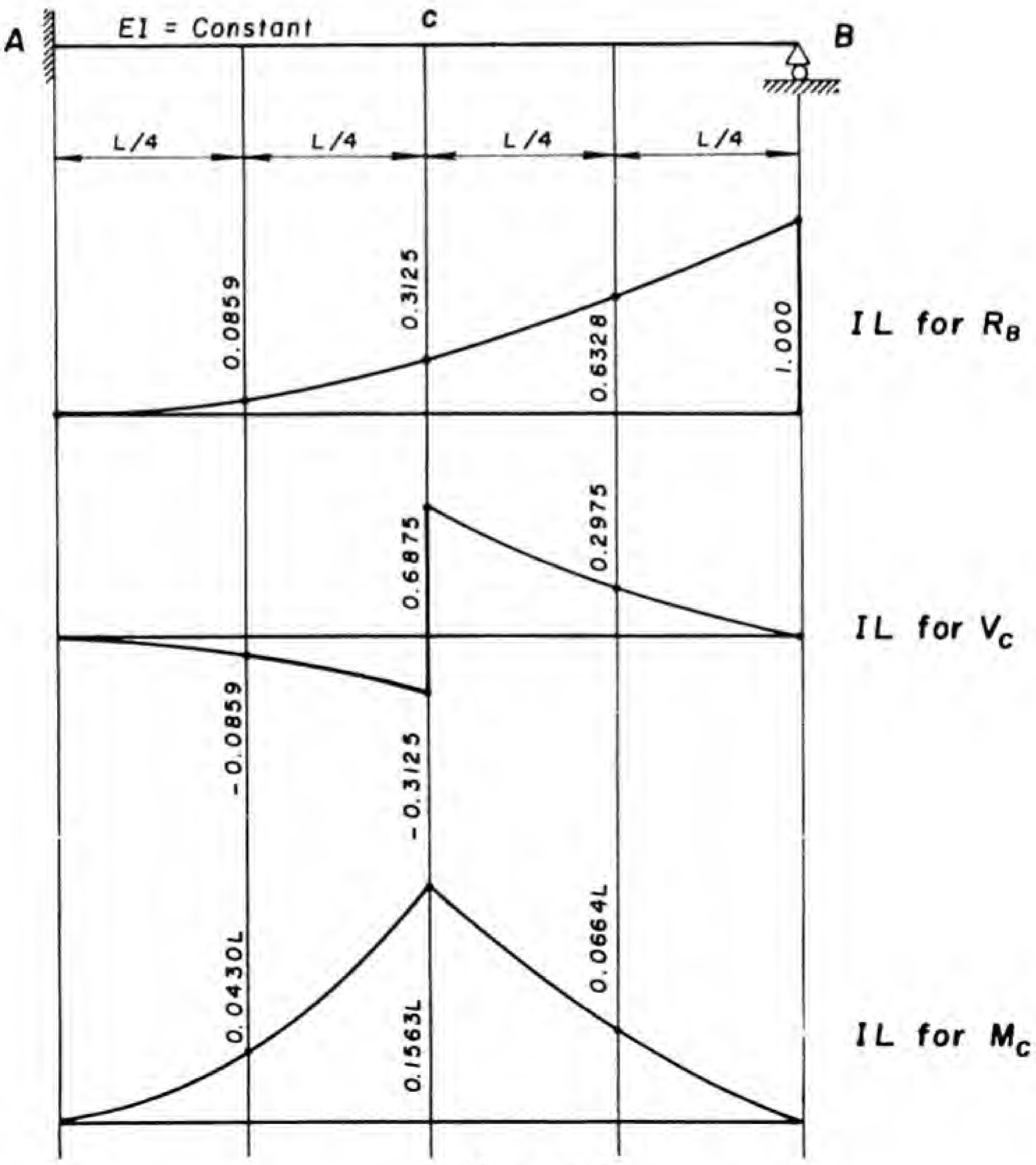


Figure 6.2

In this case the actual loading on the beam which produces the displacement δ_{bo} is the unit load $P = 1$ at the movable point n, so that the second term can be written as δ_{bn}. Hence

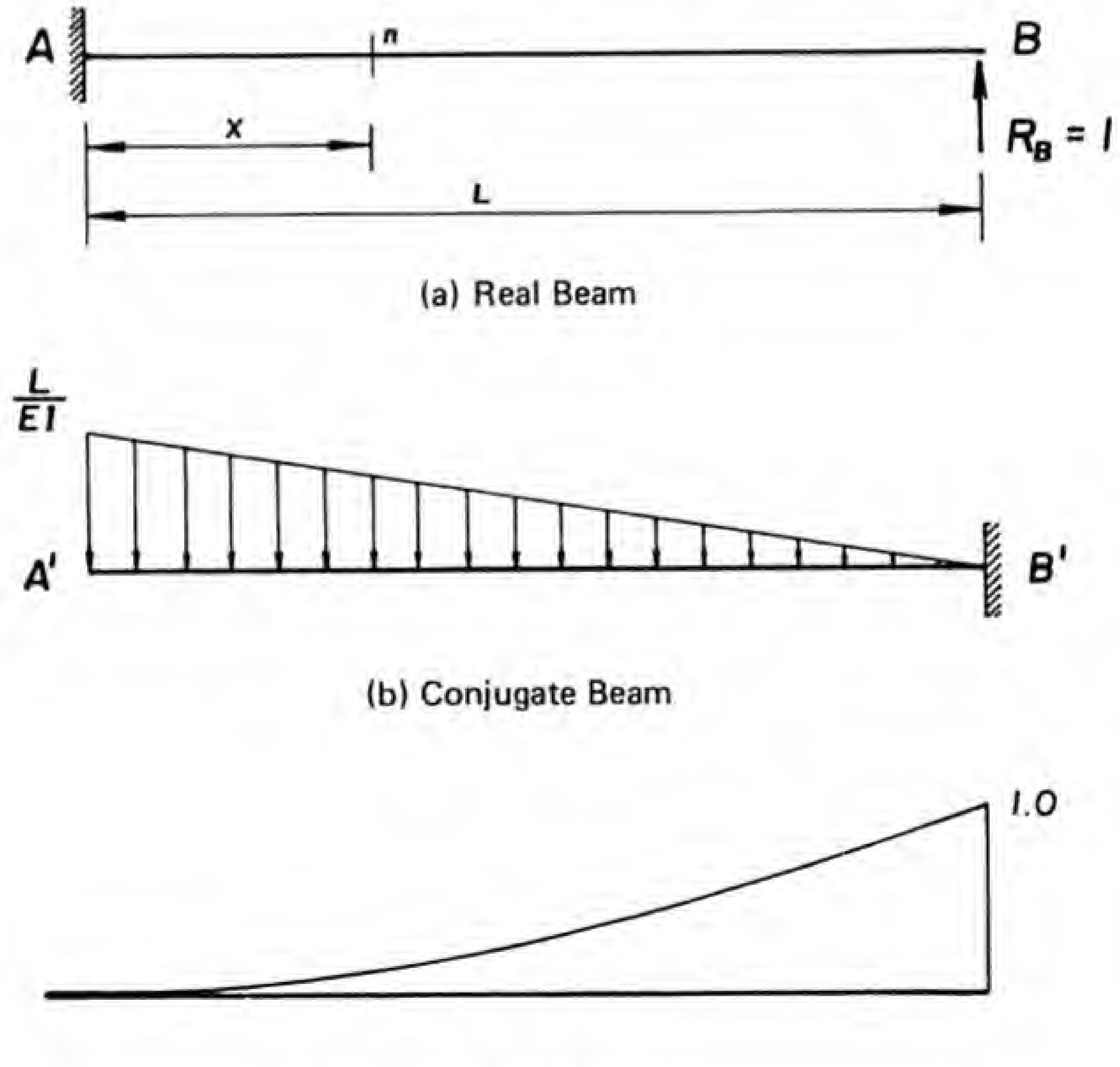
$$R_B = -\frac{\delta_{bn}}{\delta_{bb}} \qquad [6.4]$$

Using Maxwell's Reciprocal Theorem by replacing the displacement δ_{bn} by δ_{nb},

$$R_B = -\frac{\delta_{nb}}{\delta_{bb}} \qquad [6.5]$$

The above equation forms the basis for determining the ordinates of influence lines since the displacements in both the numerator and denominator are due to the same unit vertical load acting at B. The value of δ_{nb} represents the displacement in the direction of R_B due to a unit load $P = 1$ travelling along the beam. The variation of δ_{nb} represents the elastic curve of the beam due to a unit load at B in the direction of R_B. Any of the methods of displacement computation may be used to find the shape of the elastic curve of the beam under a unit load at B. By dividing the ordinate of this curve by a constant factor $(-\delta_{bb})$ gives the ordinates of the graph representing R_B when a unit load $P = 1$ traverses the beam. This curve by definition describes the influence line of R_B.

The elastic curve δ_{nb} may be obtained using the conjugate beam method as shown in Fig. 6.3.



(c) IL for R_B

Figure 6.3

The displacement at any point n due to a unit load at B is

$$\delta_{nb} = -\frac{1}{EI}\left[\left(\frac{Lx}{2}\right)\left(\frac{2x}{3}\right) + \frac{(L-x)x}{2}\left(\frac{x}{3}\right)\right]$$

$$= -\frac{x^2}{6EI}(3L - x)$$

In a similar manner the displacement at B due to $R_B = 1$ is

$$\delta_{bb} = \frac{L^3}{3EI}$$

Hence, the equation of the influence line for the reaction at B becomes

$$R_B = -\frac{\delta_{nb}}{\delta_{bb}} = \frac{x^2(3L - x)}{2L^3}$$

The kinematic method may also be used to construct the influence line for internal stress resultants such as moments and shears. For the propped cantilever beam, the bending moment at midspan may be taken as the redundant reaction moment by introducing a hinge as shown in Fig. 6.4 if it is required to draw the influence line of the moment at midspan.

The redundant reaction moment is determined from compatibility conditions at the hinge such that the change of rotation of the two continuous sections to the right and to the left of the hinge must be zero. Thus

$$M_C\delta_{cc} + \delta_{nc} = 0$$

or

$$M_C = -\frac{\delta_{nc}}{\delta_{cc}} \qquad [6.6]$$

Therefore, the influence line for the bending moment at midspan will have the same shape as the deflection of the beam with a hinge at midspan. The conjugate beam method will be sufficient to determine the moment at C, which will be equal to δ_{cc} and the deflections at selected points along the span. For example, at point n $(0 < x < L/2)$

$$\delta_{nc} = \text{moment at } n \text{ of the conjugate beam}$$

$$= \frac{1}{EI}\left(\frac{2(L-x)x^2}{2L} + \frac{2x^3}{3L}\right)$$

$$= \frac{x^2}{EIL}\left(\frac{3L - x}{3}\right)$$

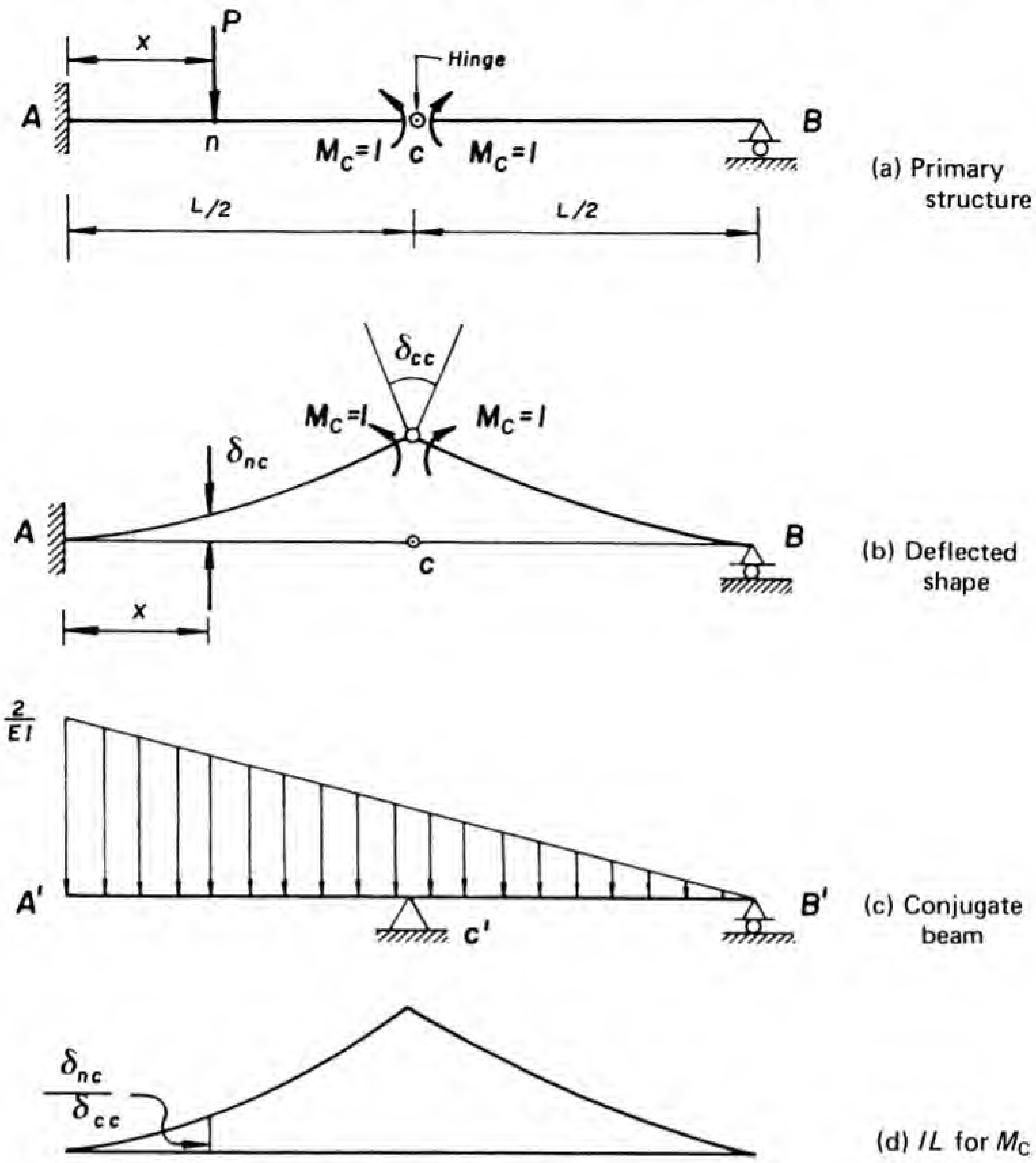


Figure 6.4

$$\delta_{cc} = \text{Reaction at C of the conjugate beam}$$

$$= \frac{1}{EI}\left[(2)(L)\left(\frac{1}{2}\right)\left(\frac{2}{3}L\right)\right] \Big/ \left(\frac{L}{2}\right)$$

$$= \frac{4L}{3EI}$$

The bending moment at point n is

$$M_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= \frac{x^2(3L - x)}{4L^2} \text{ for } 0 < x < L/2$$

Similarly, when $x > L/2$

$$\delta_{nc} = \frac{x^2}{EIL}\left(\frac{3L - x}{3}\right) - \frac{4L}{3EI}(x - L/2)$$

The corresponding bending moment is

$$M_C = \frac{x^2(3L - x) - 4L^2x + 2L^3}{4L^2} \quad \text{for } L/2 < x < L$$

Notice that the above expressions are identical with those obtained by statical method.

When using the kinematic method to determine the influence line for the shear at the midspan of the propped cantilever beam, the beam may be made determinate by first cutting at C and inserting a two-bar linkage as shown in Fig. 6.5. This linkage system cannot carry shear force and distorts into a parallelogram shape thus establishing an equal rotation of the left and right tangents at C.

The displacement of the beam at point n $(0 < x < L/2)$ determined from the conjugate beam is

$$\delta_{nc} = -\frac{1}{EI}\left[(L - x)\left(\frac{x^2}{2}\right) + \left(\frac{x^2}{2}\right)\left(\frac{2x}{3}\right)\right]$$

$$= -\frac{x^2}{6EI}(3L - x) \quad \text{for } 0 < x < L/2$$

$$\delta_{cc} = M'_C = \frac{1}{EI}\left[(L)\left(\frac{L}{2}\right)\left(\frac{2L}{3}\right)\right] = \frac{L^2}{3EI}$$

The shear force when the load is at point n is

$$V_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= -\frac{x^2(3L - x)}{2L^3}$$

Similarly, when $x > L/2$

$$\delta_{nc} = L^3/3EI - \frac{x^2}{6EI} - \frac{x^2}{6EI}(3L - x)$$

The corresponding shear force is

$$V_C = \frac{\delta_{nc}}{\delta_{cc}}$$

$$= 1 - \frac{x^2(3L - x)}{2L^3} \quad \text{for } L/2 < x < L$$

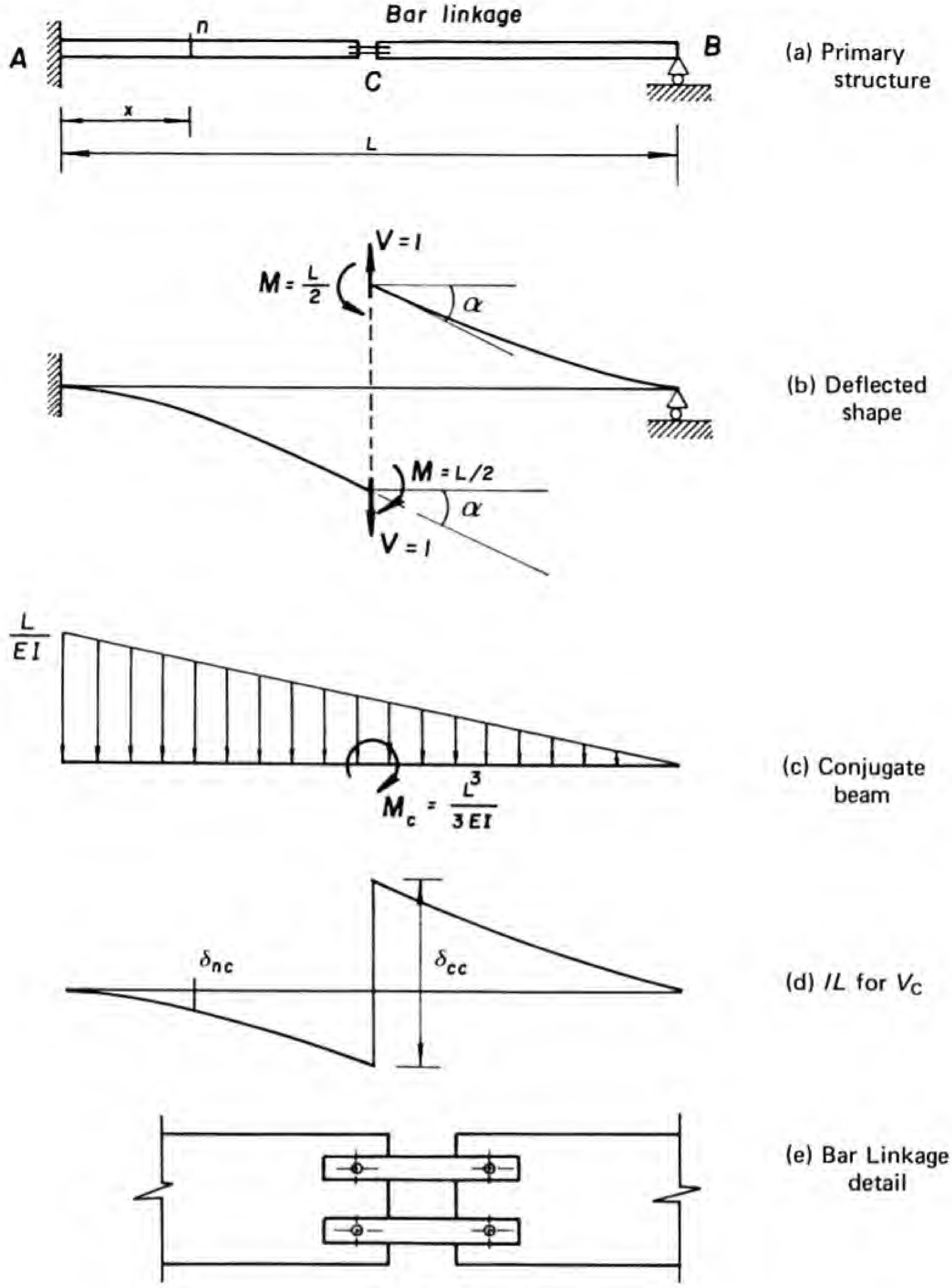


Figure 6.5

It may be concluded, therefore, that the kinematic method permits a simplified approach for the determination of the shape of the influence line for any action. This shape is identical to that of the elastic curve of the corresponding primary structure loaded by a unit force or moment at the point the redundant has been removed. This analogy, first recognised by Müller-Breslau, provides the most widely used and convenient method of computing the influence lines of indeterminate structures. The principle of Müller-Breslau may be stated as follows:

The ordinates of the influence lines of any function (such as reaction, moment,

shear) of any structure is represented, to some scale, to those of the displacement curve which is obtained by removing the restraint corresponding to the relevant function imposing in its place a unit distortion such as rotation or a linear displacement.

6.3 INFLUENCE LINES FOR MULTIPLE REDUNDANT STRUCTURES

6.3.1 Influence Line for Bending Moment

Applying the Müller-Breslau principle to construct the influence line for moment at any point E between the support B and C of Fig. 6.6(a), a hinge is inserted at E so that the moment capacity of the beam is removed without impairing its shear capacity.

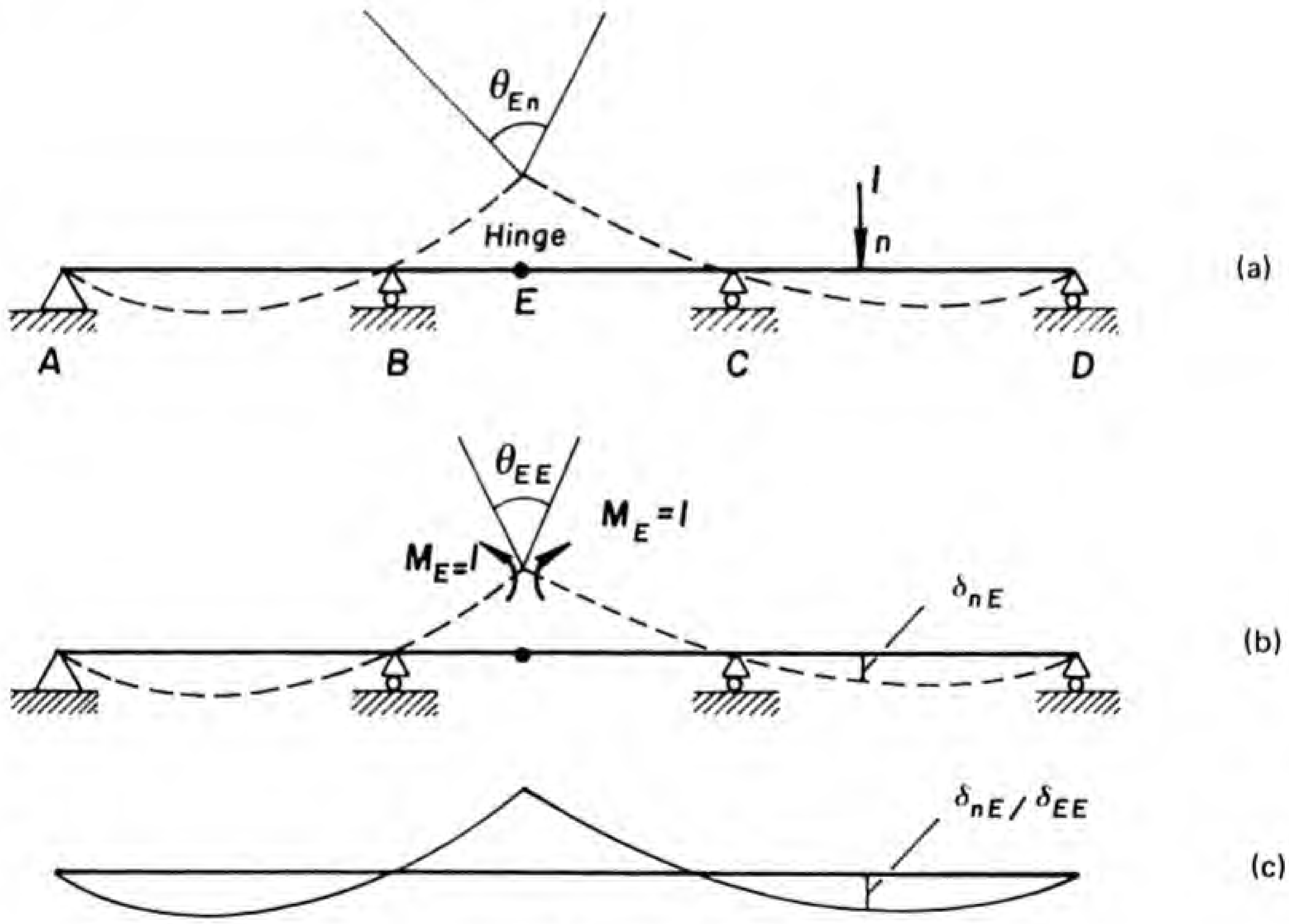


Figure 6.6

The beam is subjected to two force systems as shown in Fig. 6.6(b). A unit load is applied at *n* with the beam deflecting as shown in Fig. 6.6(a). The unit load is removed, and equal and opposite moments *M* are applied on either side of the hinge. The deflected shape of the beam is shown in Fig. 6.6(c). The shape of the deflected beam is to some scale the influence line of M_E. Thus

$$M_E = \frac{\theta_{En}}{\theta_{EE}} = \frac{\delta_{nE}}{\theta_{EE}} \quad [6.7]$$

If $\theta_{EE} = 1$ radian, then

$$M_E = \delta_{nE} \qquad [6.8]$$

Thus the influence line for M_E is obtained by dividing the ordinates of the deflected shape by θ_{EE} or by setting θ_{EE} equal to unity.

6.3.2 Influence Line for Shear Force

Let it be required to draw the influence line for shear at point E of the beam of Fig. 6.7(a). The beam is cut at E such that the shearing resistance is removed without impairing the flexural resistance. The beam is cut at E and a linkage system or slide device is inserted which cannot carry shear force and thus which permits a relative transverse deflection without introducing a change in slope of the left and right tangents at E.

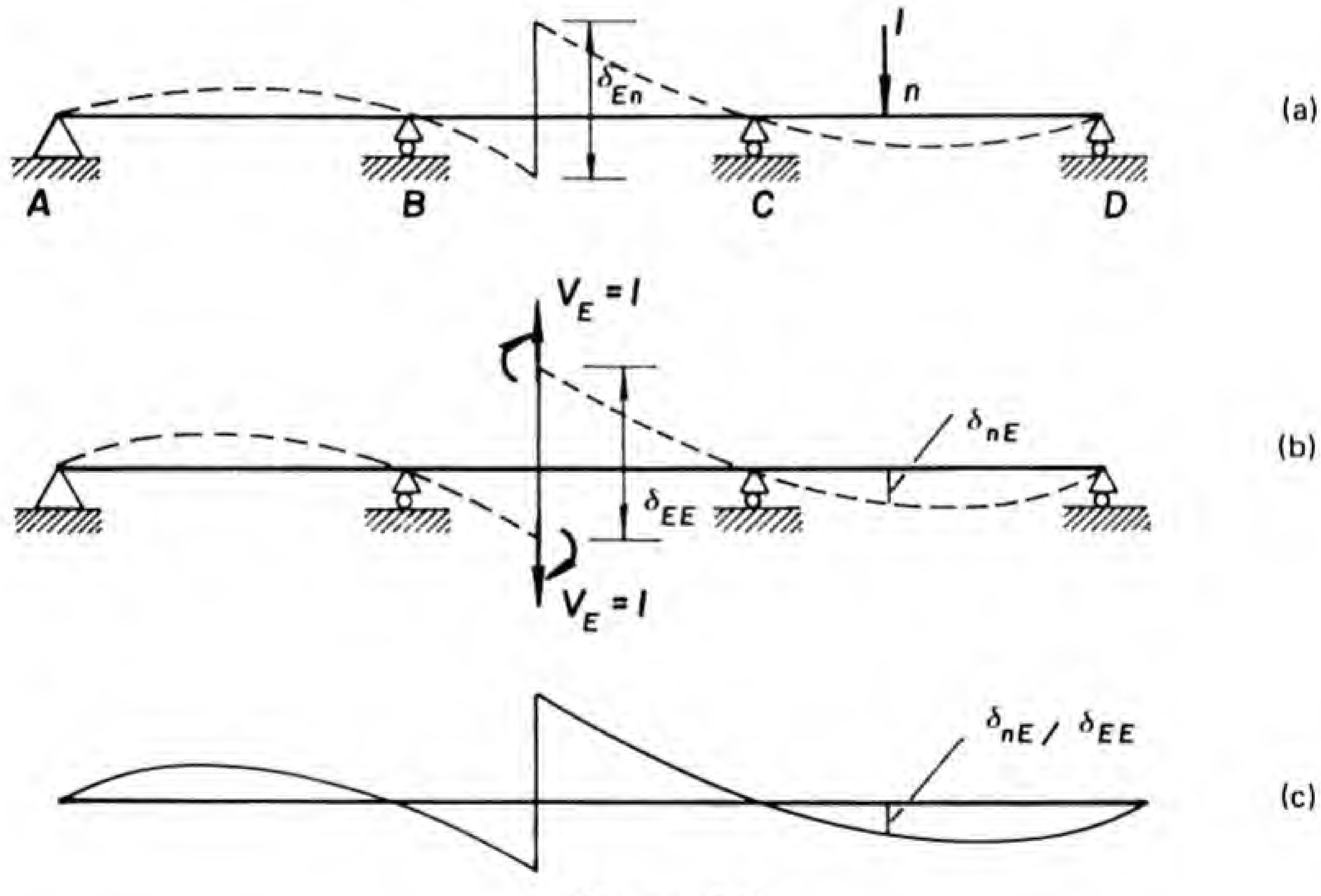


Figure 6.7

The beam is subjected to a unit load at n resulting with a vertical displacement δ_{En} at E. After removing the unit load, a pair of unit loads are applied as shown in Fig. 6.7(b). The shear force at E is given by

$$V_E = \frac{\delta_{En}}{\delta_{EE}} = \frac{\delta_{nE}}{\delta_{EE}} \qquad [6.9]$$

If δ_{EE} = unity, then

$$V_E = \delta_{nE} \qquad [6.10]$$

Therefore, the influence line for V_E is obtained by dividing the ordinates of the deflected shape by δ_{EE}.

EXAMPLE 6.1 *Compute ordinates of the influence line for the moment at B of the beam shown in Fig. 6.8. Use intervals of 2.5 m for span AB and 4.0 m for span BC. The moment of inertia is constant.*

Using the Müller-Breslau Principle, the capability of the beam to resist moment at the section for which the moment influence line is desired is removed by introducing a pin. Unit moments, say in kN m, are applied to the beam at each side of the pin at B. The modified beam, which will deflect as indicated by the dashed line, is shown in Fig. 6.8(b). According to the Müller-Breslau Principle, the various influence line ordinates are computed from the relation

$$M_B = \frac{\delta_{DB}}{\delta_{DD}}$$

Thus, the values of δ_{BB}, as well as the deflections at the necessary sections of the modified beam, may be evaluated using the conjugate beam method.

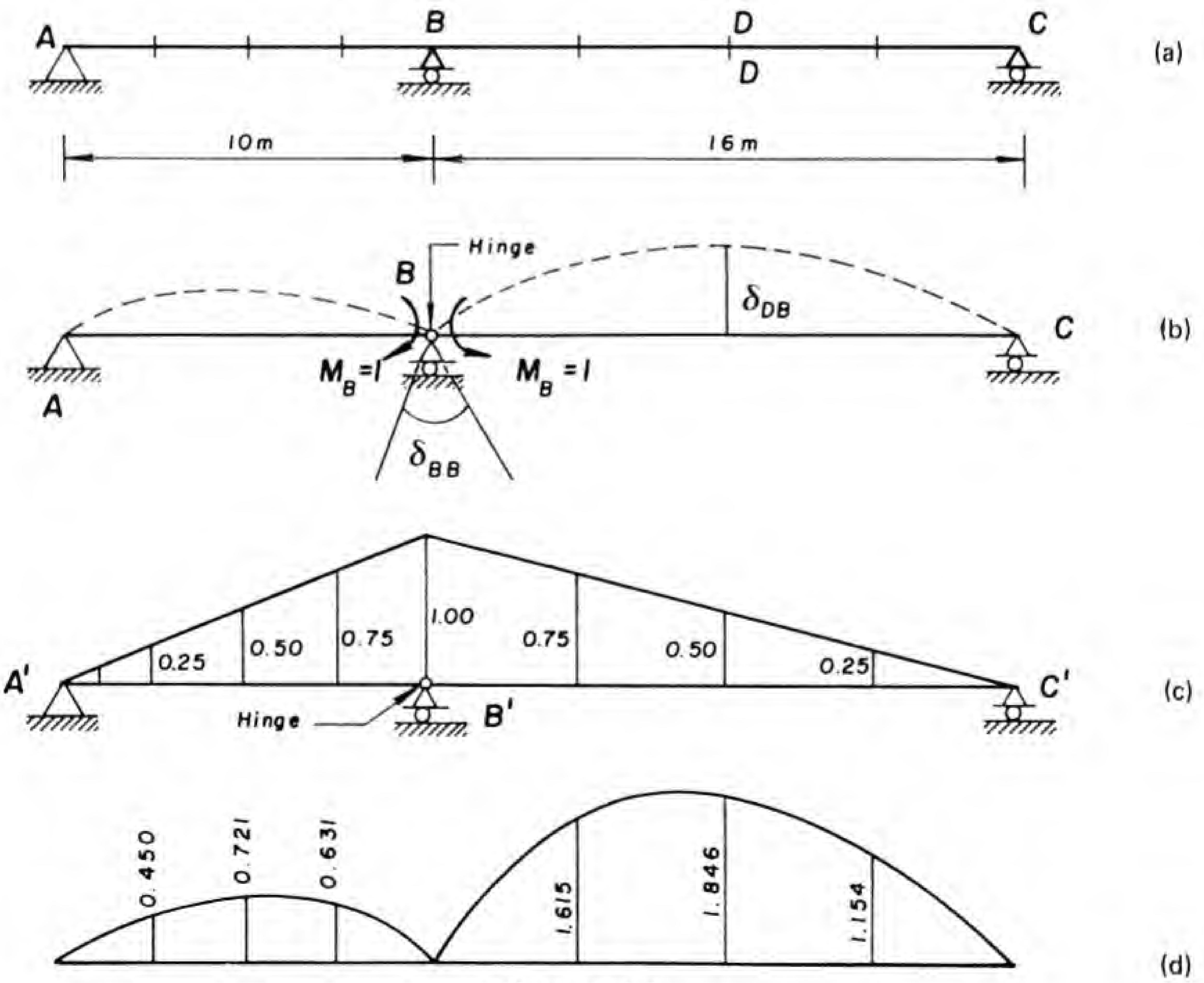


Figure 6.8

Determination of the reactions (*slopes*):

$$R_A = \frac{1}{3}\left(\frac{10 \times 1}{2}\right) = 1.667$$

$$R_B = \frac{2}{3}\left(\frac{10 \times 1}{2} + \frac{16 \times 1}{2}\right) = 8.667 = \delta_{BB}$$

$$R_C = \frac{1}{3}\left(\frac{16 \times 1}{2}\right) = 2.667$$

Determination of moments (*deflections*):

$$M_{2.5} = 1.667 \times 2.5 - \left(\frac{2.5 \times 0.25}{2}\right)\left(\frac{2.5}{3}\right) = 3.906$$

$$M_{5.0} = 1.667 \times 5.0 - \left(\frac{5.0 \times 0.50}{2}\right)\left(\frac{5.0}{3}\right) = 6.250$$

$$M_{7.5} = 1.667 \times 7.5 - \left(\frac{7.5 \times 0.75}{2}\right)\left(\frac{7.5}{3}\right) = 5.469$$

$$M_{10} = 0$$

$$M_{14} = 2.667 \times 12 - \left(\frac{12.0 \times 0.75}{2}\right)\left(\frac{12}{3}\right) = 14.00$$

$$M_{18} = 2.667 \times 8 - \left(\frac{8 \times 0.50}{2}\right)\left(\frac{8}{3}\right) = 16.00$$

$$M_{22} = 2.667 \times 4 - \left(\frac{4 \times 0.25}{2}\right)\left(\frac{4}{3}\right) = 10.00$$

The value of the influence line ordinate at each point is determined by dividing each moment by $\delta_{BB} = 8.667$. The resulting influence line is shown in Fig. 6.8(d).

The method of solution is first to assume that the fixed-end moment of 100 kN m exists at support B of the member BA with no other fixed-end moments being considered to exist.

6.3.3 Influence Lines by Moment Distribution

The Cross method of moment distribution may be used to obtain more easily influence lines for continuous beams and frames. The method is illustrated by the following example.

EXAMPLE 6.2 *Compute the ordinates of the influence line for the moment at B of Example 6.1 using the moment distribution method.*

The method of solution is first to assume that the fixed-end moment of 100 kN m exists at support B of the member BA with no other fixed-end moments being considered to exist. Using the moment distribution method, the moments are obtained from the balancing operation. Next, a fixed moment of 100 kN m is assumed at end B of member BC and again a similar operation is performed.

The moment distribution is shown in Table 6.2

Table 6.2

Joint	A	B		C
Member	AB	BA	BC	CB
K	$I/10$	$I/10$	$I/16$	$I/16$
DF	1.0	0.6154	0.3846	1.0
M^F_{AB} = 100	0	+100 −61.54	0 −38.46	0
	Σ	+38.46	−38.46	
M^F_{BC} = 100	0	0 −61.54	+100 −38.46	0
	Σ	−61.54	+61.54	

After finding the final moments due to the 100 kN m moment at every point where fixed-end moments can exist, the equation for M_{BA} in terms of the initial fixed-end moments is

$$M_{BA} = 0.3846M^F_{BA} + 0.6154MF^F_{BC}$$

The fixed-end moments for a 1.0 kN load placed successively at each of the points for which an influence line ordinate is desired, are computed below. The fixed-end moment for a propped cantilever loaded with one concentrated load P is

$$M^F_{BA} = \frac{Pab}{L^2}\left(a + \frac{b}{2}\right)$$

where a is measured from the pinned-end.

For span AB:

Table 6.3

$x = a$	$b = 10 - a$	$M^F_{AB} = ab(a + b/2)/L^2$
2.5	7.5	1.172
5.0	5.0	1.875
7.5	2.5	1.641

For span BC:

Table 6.4

$x = a$	$b = 16 - a$	$M^F_{BA} = ab(a + b/2)L^2$
4.0	12.0	1.875
8.0	8.0	3.000
12.0	4.0	2.625

Note: The ordinate x is measured with support C as the origin. The above values of the fixed moments are substituted in the equation above for M_{BA} and the influence line ordinates are computed in Table 6.5.

Table 6.5

Point, x(m)	M^F_{BA}	M^F_{BC}	$M_{AB} = 0.3846M^F_{BA} + 0.6154M^F_{BC}$
0	0		0
2.5	1.172		0.450
5.0	1.875		0.721
7.5	1.641		0.631
10.0		0	0
14.0		2.625	1.615
18.0		3.000	1.846
22.0		1.875	1.154
26.0			

The influence line for the moment at support B is shown in Fig. 6.8(d).

6.4 PROBLEMS

6.1 Draw the influence lines for the beam shown in Fig. P6.1 for the support reactions R_A and R_B.

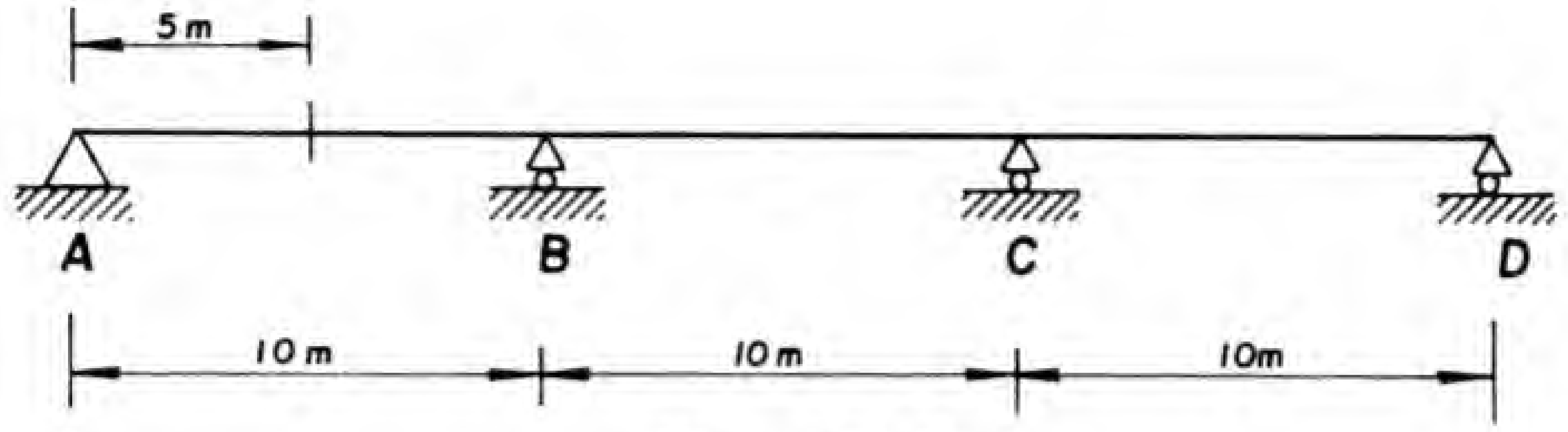


Figure P6.1

6.2 Draw the influence lines for the beam shown in Fig. P6.1 for the moments M_B and M_5.

6.3 Draw the influence lines for the beam shown in Fig. P6.1 for the shear S_5.

7. Introduction to Matrix Analysis

7.1 INTRODUCTION

After the introduction of high-speed computers, there has been a revolution in structural analysis, not only in the computational methods but also in the fundamental theorems. Since digital computers are ideally suitable for automatic computations of matrix algebra, it was found desirable to formulate the entire structural analysis in matrix notation. Matrix methods of structural analysis are based on the concept of replacing the actual structure by an equivalent analytical model consisting of discrete structural elements having known properties which can be expressed in matrix form. Matrices are useful in expressing structural theory and in producing an efficient means for carrying out numerical calculations.

Two methods have been formulated in matrix structural analysis: the flexibility and stiffness methods. It will not be possible in this textbook to develop the two matrix methods to sufficient depth. The methods are developed to the level of manual computation.

7.2 FORCE AND DISPLACEMENT MEASUREMENTS

It is evident that the overall description of the behaviour of a structure is accomplished through the dual consideration of force and displacement components at designated points. There are a number of ways of measuring a force applied to a structure or its displacement at designated points in a prescribed direction. Such points are commonly known as *node points*. The first step in the analysis of structures is to idealise the actual structure into a mathematical model which consists of distinct structural elements interconnected through node points. In this text the word *force* includes moment.

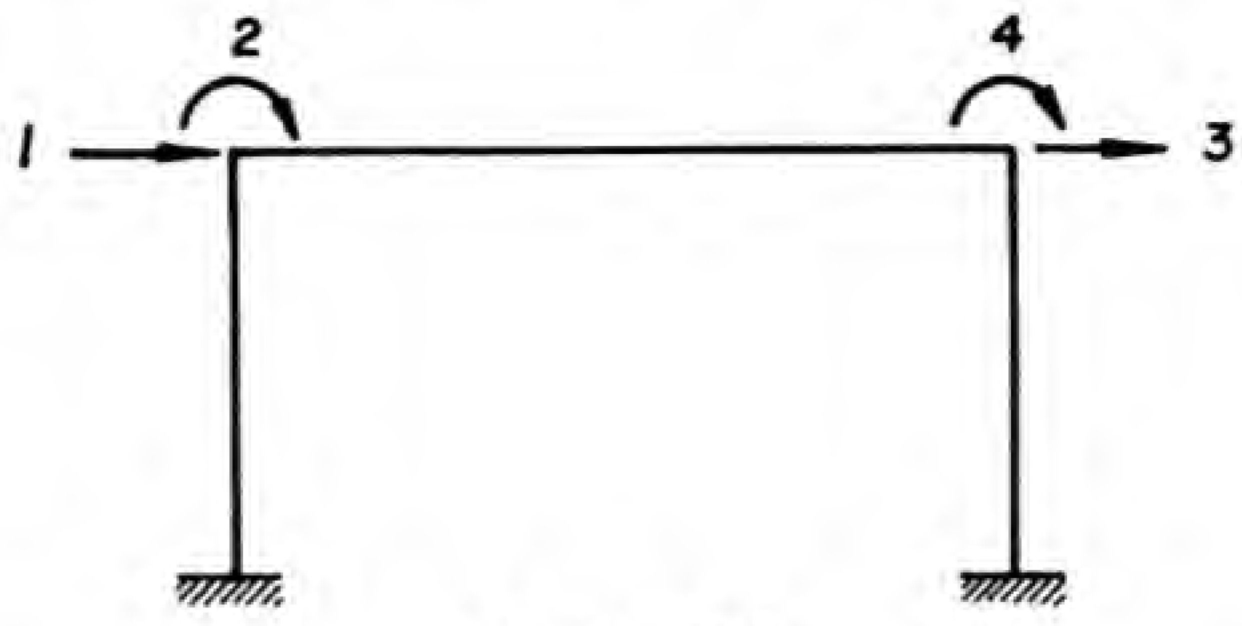


Figure 7.1

To designate the forces and displacements at the nodes of a given structure, a *coordinate system* is used to identify these measurements. For the frame shown in Fig. 7.1, for example, the system consists of four arbitrary coordinates which are identified by four numbered arrows shown at the specific nodes or joints. The forces are listed in column matrix [P] and is referred to as a *force vector* and represents an ordered array of force measurements. For instance, the force vector for the frame of Fig. 7.1 is represented by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \tag{7.1}$$

Likewise, the coordinate displacement vector, having the same significance as in the force vector may be expressed as

$$[\Delta] = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{7.2}$$

In a similar manner, the forces and displacements at the nodes of a given element may be designated by listing in column matrices [P] and [Δ], respectively. For the beam element of Fig. 7.2, for example, with direct forces at

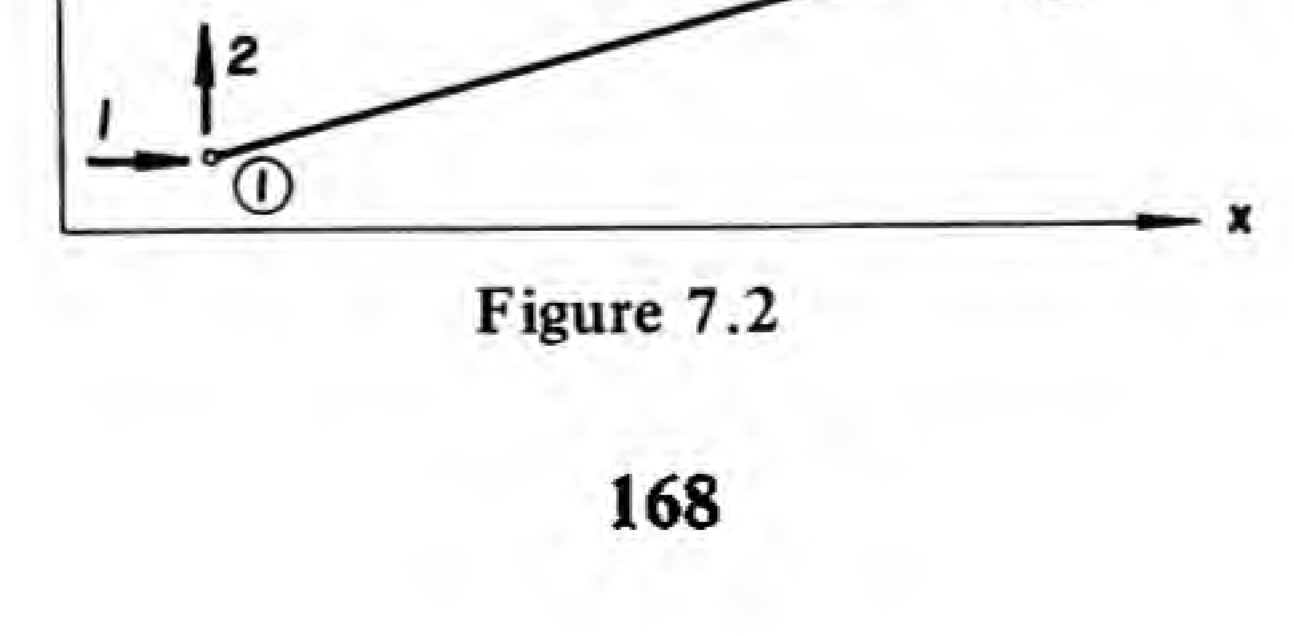


Figure 7.2

node 1 and moments at node 2, the force vector is written as

$$[P] = \begin{bmatrix} P_{x1} \\ P_{y1} \\ M_{x2} \\ M_{y2} \end{bmatrix} \qquad [7.3]$$

and the displacement vector as

$$[\Delta] = \begin{bmatrix} u_1 \\ v_1 \\ \theta_{x2} \\ \theta_{y2} \end{bmatrix} \qquad [7.4]$$

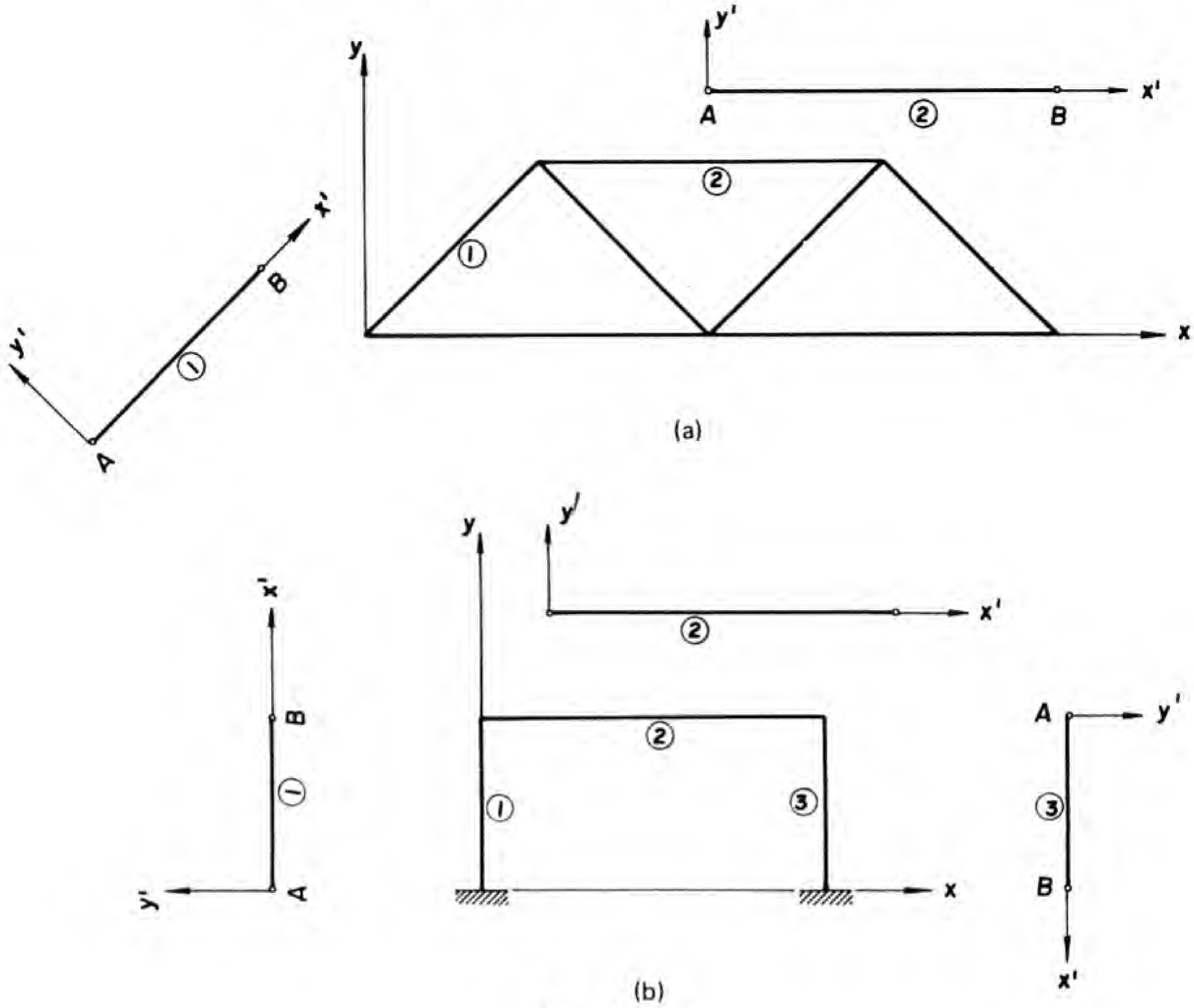
A necessary step in the formation of the force and displacement vectors is the establishment of the node points and their location with respect to coordinate axes. At this stage it is necessary to define two sets of orthogonal coordinate systems. The first set is that of the structure, known as the *global axes*, and consists of a single coordinate system. The second set is that of the members or elements, known as the *local axes*, and consists of one coordinate system for each member. Since the members are in general differently oriented within a structure, these axes originating at member ends will usually be differently oriented from one element to the next. Global and local coordinates are illustrated in Fig. 7.3(a) for trusses and in Fig. 7.3(b) for frames.

When forces are applied to structures, displacements occur. Alternatively, when displacements are prescribed, node forces are necessary to produce them. The relationships that exist between applied forces and displacements play an important role in structural analysis. The force and displacement characteristics of a structure are usually described under definitions of *flexibility* and *stiffness* coefficients. The flexibility and stiffness coefficients depend on the force-displacement properties of the structure and the coordinate system used.

A simple illustration of such relationships is obtained by considering a linear elastic spring shown in Fig. 7.4. Single coordinate is indicated for the force and displacement measurements. The force P will stretch the spring thereby producing a displacement Δ at the end of the spring. The relationship between P and Δ can be expressed as

$$\Delta = fP \qquad [7.5]$$

In [7.5], f is the *flexibility coefficient* of the spring and is defined as the value of the displacement at node 1. In general, a flexibility coefficient is the value of the displacement at a point of the structure, in a given direction, due to a unit force applied at a second point in a second direction.



(b)

Figure 7.3

An alternative way is to establish a relationship between the force P and the displacement Δ for the spring of Fig. 7.4. The force P required to produce a displacement Δ units is determined from

$$P = k\Delta \tag{7.6}$$

In [7.6], k is the *stiffness coefficient* of the spring and is defined as the value of the force required at coordinate 1 to produce a unit displacement at 1. In general, a stiffness coefficient is the value of the force at a point of the structure, in a given direction, due to unit displacement applied at a second point in a second direction.

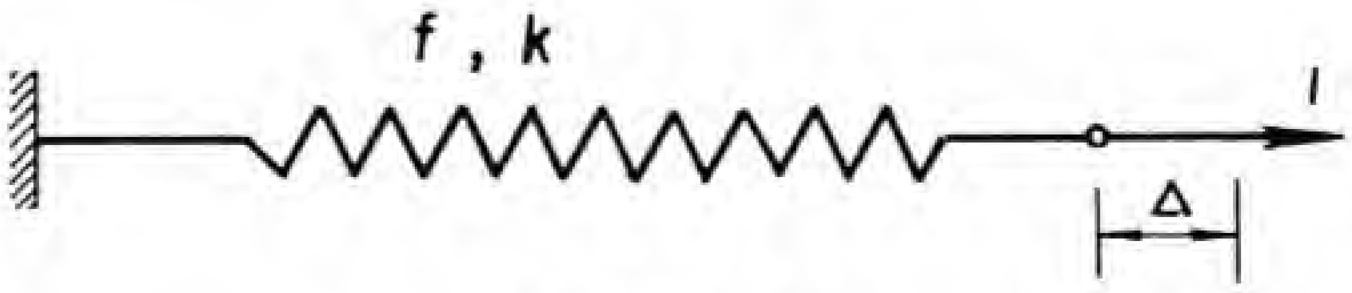


Figure 7.4

Comparison of [7.5] and [7.6] reveals that the flexibility and the stiffness of the spring are *inverse* to one another.

$$f = \frac{1}{k} = k^{-1}$$
$$k = \frac{1}{f} = f^{-1} \qquad [7.7]$$

Now consider a more general case consisting of an elastic structure, supported against rigid-body motion, and subjected to loads $P_1, P_2, \ldots, P_n$ acting at nodes $1, 2, \ldots, n$. The corresponding set of displacements is represented by Δ_1, $\Delta_2, \ldots, \Delta_n$. For linearly elastic systems, the principle of superposition is applicable. Therefore, the displacement Δ_i at node i is given by

$$\Delta_i = f_{i1}P_1 + f_{i2}P_2 + \ldots + f_{in}P_n \qquad [7.8]$$

or more generally,

$$\Delta_i = \sum_{j=1}^{j=n} f_{ij}P_j \qquad [7.9]$$

By definition, f_{ij} is the displacement produced at node i due to a unit load at node j ($P_j = 1$). The coefficients f_{ij}, which are the displacements due to unit loads, are known as flexibility coefficients.

In general, for n nodes, there will be n such displacements which may be written in a single matrix equation

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} \qquad [7.10]$$

and which can be written in compact matrix form as

$$[\Delta] = [F]\,[P] \qquad [7.11]$$

where $[\Delta]$ is the column displacement matrix, $[F]$ is a square flexibility matrix and $[P]$ is a column load matrix (load vector). This equation is of the same type as [2.17].

Using matrix operation, one can solve the set of algebraic equations represented in [7.10] for forces in terms of displacements. In matrix notation

$$[P] = [F]^{-1}\,[\Delta] \qquad [7.12]$$

where $[F]^{-1}$ is the inverse of matrix $[F]$. It is noted that [7.12] has the same form as [7.6] since it expresses forces in terms of displacements. Consequently,

$$[F]^{-1} = [K] \tag{7.13}$$

where $[K]$ is the stiffness matrix which is the inverse of the flexibility matrix. Thus

$$[P] = [K]\,[\Delta] \tag{7.14}$$

The expanded form of [7.14] is

$$\begin{bmatrix} P_1 \\ P_2 \\ . \\ . \\ . \\ P_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ . \\ . \\ . \\ \Delta_n \end{bmatrix} \tag{7.15}$$

By definition, k_{ij} is the force required at node i to produce a unit displacement at node j only (zero displacements at all other nodes).

Flexibility coefficients for linear elastic behaviour have the property of reciprocity which may be expressed analytically as

$$f_{ij} = f_{ji} \tag{7.16}$$

This equation defines symmetry of $[F]$. Since $[F]$ is symmetrical the inverse of a symmetric matrix will also become symmetrical. Therefore, [7.13] guarantees that the stiffness matrix $[K]$ will likewise be symmetrical. Consequently,

$$k_{ij} = k_{ji} \tag{7.17}$$

To illustrate these matrices consider a simple cantilever beam of uniform cross section shown in Fig. 7.5(a). To determine the flexibility matrix, the influence coefficients must be determined by applying unit loads to the free end.

Due to axial load $N = 1$ (Fig. 7.5(b))

$$\begin{aligned} \delta_n &= \frac{L}{EA} \\ \delta_{vn} &= 0 \\ \theta_n &= 0 \end{aligned} \tag{7.18}$$

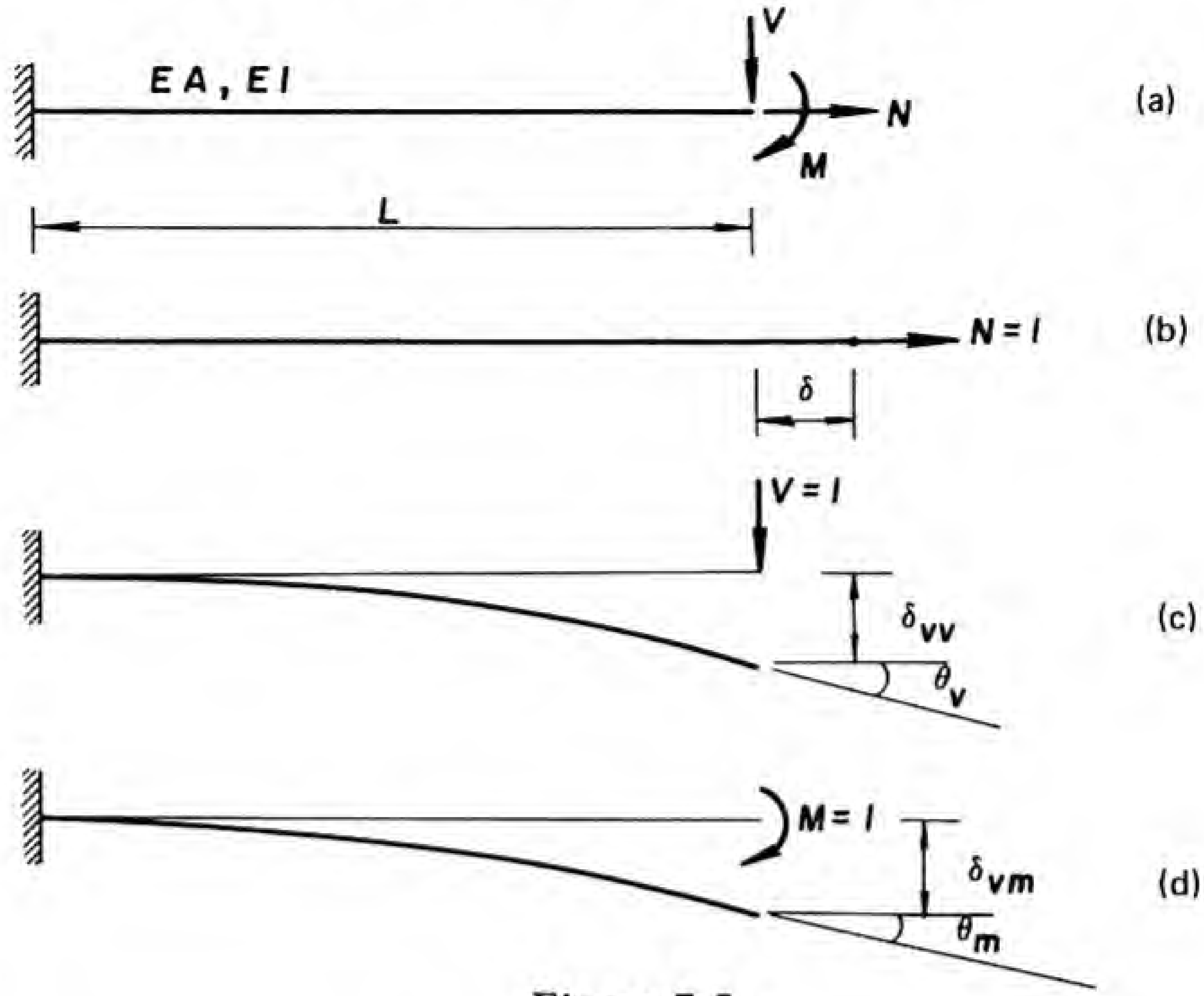


Figure 7.5

Due to vertical load $V = 1$ (Fig. 7.5(c))

$$\delta_n = 0$$

$$\delta_{vv} = \frac{L^3}{3EI} \qquad [7.19]$$

$$\theta_v = \frac{L^2}{2EI}$$

Due to moment $M = 1$ (Fig. 7.5(d))

$$\delta_n = 0$$

$$\delta_{vm} = \frac{L^2}{2EI} \qquad [7.20]$$

$$\theta_m = \frac{L}{EI}$$

The above results may be written in matrix form as

$$\begin{bmatrix} \delta_n \\ \delta_v \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{bmatrix} N \\ V \\ M \end{bmatrix} \qquad [7.21]$$

or, when written in compact matrix form

$$[\Delta] = [F]\,[P] \tag{7.11}$$

In a similar manner the stiffness matrix may be determined by unit displacements as shown in Fig. 7.6.

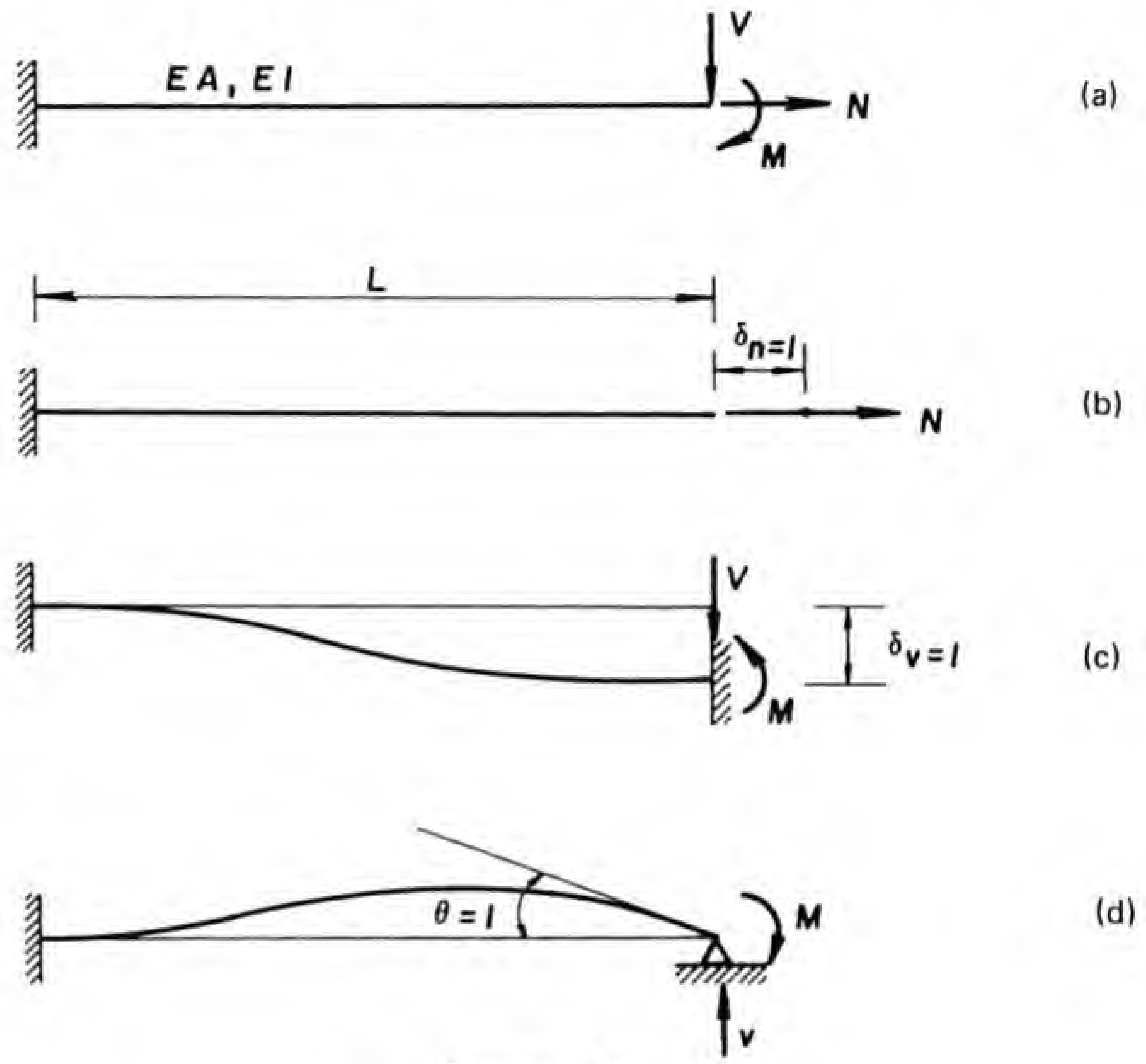


Figure 7.6

Due to unit axial displacement (Fig. 7.6(b))

$$N = \frac{EA}{L} \tag{7.22}$$

Due to unit vertical displacement (Fig. 7.6(c))

$$V = \frac{12EI}{L^3} \tag{7.23}$$

$$M = -\frac{6EI}{L^2}$$

Due to unit rotation (Fig. 7.6(d))

$$\left.\begin{aligned} V &= -\frac{6EI}{L^2} \\ M &= \frac{4EI}{L} \end{aligned}\right\} \tag{7.24}$$

The above results may be written in matrix form as

$$\begin{bmatrix} N \\ \\ V \\ \\ M \end{bmatrix} = \begin{bmatrix} \dfrac{EA}{L} & 0 & 0 \\ 0 & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ 0 & -\dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix} \begin{bmatrix} \delta_n \\ \\ \delta_v \\ \\ \theta \end{bmatrix} \qquad [7.25]$$

which may be written in compact matrix form as

$$[P] = [K]\,[\Delta] \qquad [7.26]$$

The results may be checked by matrix multiplication

$$[F]\,[K] = [K]\,[F] = [I] \qquad [7.27]$$

It is noted that both $[F]$ and $[K]$ are symmetric matrices which is the consequence of the reciprocal theorem.

7.3 THE FLEXIBILITY METHOD

The basic theory of the flexibility method is developed in this section, and the concepts are clarified by numerical examples. The development of the method rests on the basic principles of equilibrium of forces, compatibility and linear force-displacement relationships.

Consider a structure, which is idealised into a model consisting of distinct structural elements interconnected through node points, under the action of generalised external forces applied at the nodes $P_1, P_2, \ldots P_n$. These may be conveniently represented by a column matrix or force vector $[P]$

$$[P] = \{P_1, P_2 \ldots P_n\} \qquad [7.28]$$

Let it be assumed that the structure consists of m redundants which are forces to be determined, that is

$$[X] = \{X_1, X_2 \ldots X_m\} \qquad [7.29]$$

which are the redundant forces or reactions. If such redundants are removed, the structure becomes determinate and the internal forces are determined from conditions of equilibrium alone. In an indeterminate structure, the internal forces must also satisfy compatibility in addition to equilibrium. In dealing with an indeterminate structure with m redundants, the redundants are treated as additional loads on the statically determinate structure. It is assumed that the structure is composed of an assemblage of j simple elements. Internal forces exist in the structure at the node points. If the internal force members are

represented by the vector $[S]$ where

$$[S] = \{S_1 \quad S_2 \quad \ldots \quad S_j\} \qquad [7.30]$$

then, $[S]$ can be related to the applied loads $[P]$ and $[X]$ as

$$\begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} = [\mathrm{B_0}] \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} + [\mathrm{B_1}] \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X \end{bmatrix} \qquad [7.31(a)]$$

which is written in compact notation as

$$[S] = [\mathrm{B_0}]\,[P] + [\mathrm{B_1}]\,[X] \qquad [7.31(b)]$$

or using partitioned matrices

$$[S] = [\mathrm{B_0} \mid \mathrm{B_1}] \begin{bmatrix} P \\ \hline X \end{bmatrix} \qquad [7.31(c)]$$

where, in general, $[B_0]$ and $[B_1]$ are rectangular matrices whose elements are obtained from equilibrium conditions of the structure. For example, if P_i is taken as a unit load with all other loads including $[X]$ held at zero, the internal forces in the structure represent the coefficients corresponding to the ith column in the $[B_0]$ matrix. Likewise, the internal forces which result from a unit load X_j with all others held at zero represent the coefficients corresponding to the jth column of the $[B_1]$ matrix.

To formulate the compatibility condition, the principle of least work will be utilised which may be stated as: *The true values of the redundant forces are those which make the strain energy U of the strained structure a minimum.*

The strain energy is given as

$$U = \frac{1}{2}\,[S_1 \quad S_2 \quad \ldots \quad S_j] \begin{bmatrix} F_1 & & & & \\ & F_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & F_j \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_j \end{bmatrix} \qquad [7.32]$$

which is written in compact matrix form as

$$U = \frac{1}{2}\,[S]^{\mathrm{T}}\,[F]\,\{S\} \qquad [7.33]$$

In order to obtain the strain energy U in terms of the unknown $\{X\}$,

substitute [7.31(b)] into [7.33]. In doing so, note that the transpose $\{S\}^T$ may be written from [7.31(b)] as

$$[S] = \begin{bmatrix} P \\ \text{--} \\ X \end{bmatrix}^T [B_0 \mid B_1]^T$$

$$= [P \mid X] [B_0 \mid B_1]^T \qquad [7.34]$$

Substituting [7.31(b)] and [7.34] into [7.33]

$$U = \frac{1}{2} [P \mid X] [H] \begin{bmatrix} P \\ \text{--} \\ X \end{bmatrix} \qquad [7.35]$$

where

$$[H] = [B_0 \mid B_1]^T [F] [B_0 \mid B_1] \qquad [7.36]$$

Since [P] and [X] are the applied and redundant forces, respectively, it is convenient to partition [H] to conform to the load vectors, thus,

$$U = \frac{1}{2} [P \mid X] \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right] \begin{bmatrix} P \\ \cdots \\ X \end{bmatrix} \qquad [7.37]$$

After expanding [7.37]

$$U = \frac{1}{2} ([P] [H_{pp}] [P] + [P] [H_{px}] [X] + [X] [H_{xp}] [P] + [X] [H_{xx}] [X]) \qquad [7.38]$$

Utilising the theorem of least work and noting that the [H] matrix is symmetric gives

$$\frac{\partial U}{\partial X} = [H_{xp}] \; [P] + [H_{xx}] [X] = 0 \qquad [7.39]$$

from which the redundants are determined as

$$[X] = -[H_{xx}]^{-1} [H_{xp}] [P] \qquad [7.40]$$

Solving for [X] from [7.40] all internal forces can be determined from [7.31].

Summarising, the essential steps in applying the flexibility method to lead to the solution of structural problems may be stated as follows:

1. Idealise the structural problem to be analysed
2. Specify the redundant forces and identify the internal member forces

3. Find the $[B_0]$ matrix for unit values of external forces; only one external force must act at a time with all other forces held at zero
4. Find the $[B_1]$ matrix for unit values of redundant forces; only one redundant force must act at a time with all other forces held at zero
5. Find the flexibility matrix $[F]$ for all members following the sequential order of the member forces in $[B_0]$ and $[B_1]$
6. Formulate the $[H]$ matrix of [7.36]
7. Calculate the redundant forces $[X]$ from [7.40]
8. Calculate the internal forces $[S]$ from [7.31]

EXAMPLE 7.1 *Using the flexibility matrix method determine the bar forces in the truss with double diagonal system shown in Fig. 7.7(a). The area of all top chord is twice the area of all remaining members.*

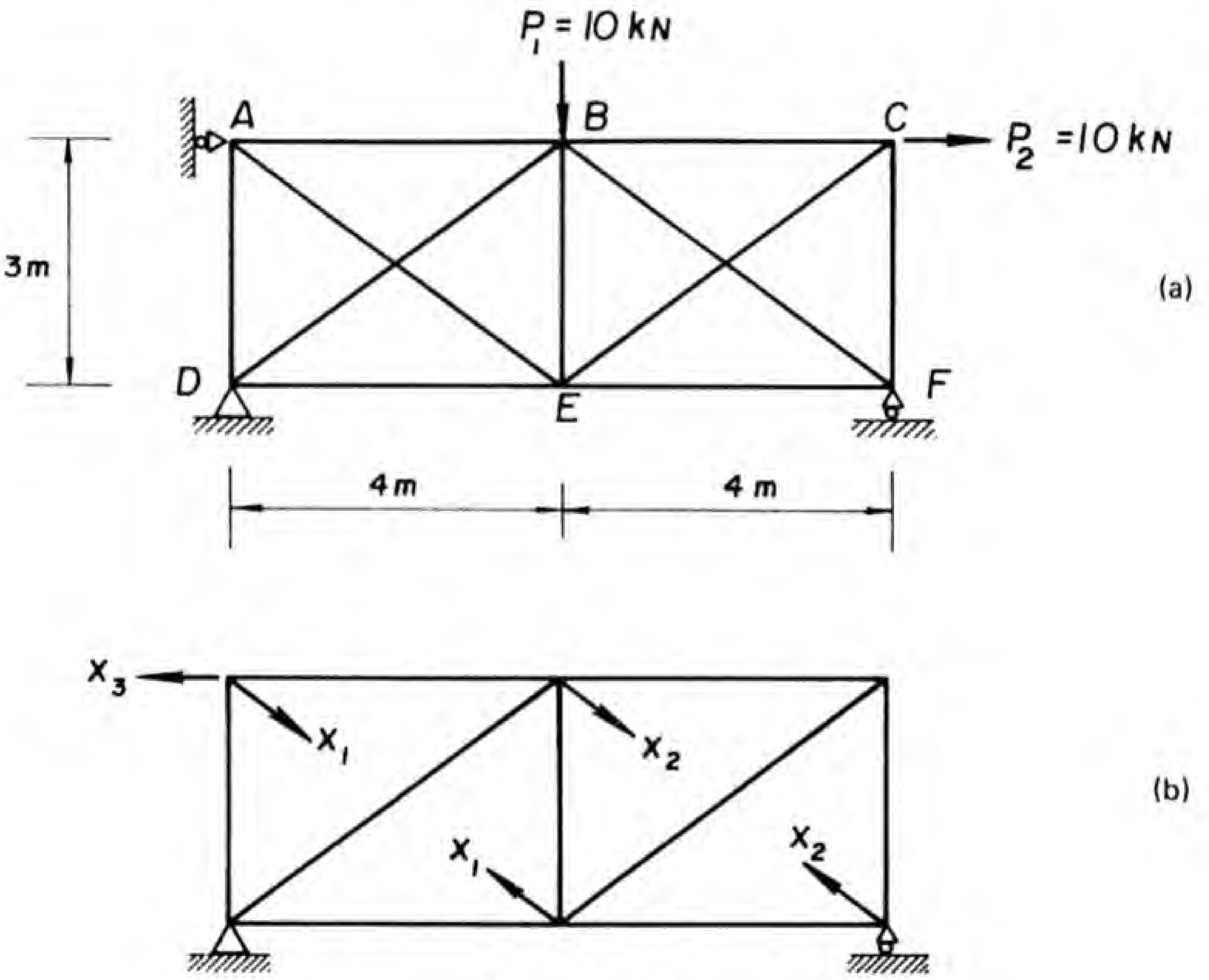


Figure 7.7

As can be easily seen, the truss is redundant to the second degree. For the selection of the redundant members several choices exist. Here members AE and BE and the reaction at A are taken as the redundants, then the truss is reduced to a determinate one as shown in Fig. 7.7(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time with all other forces including the redundants held at zero, then calculate the

internal forces in all members for each case. Thus,

$$
\begin{array}{ccc}
& \begin{array}{cc} P_1 = 1 & P_2 = 1 \end{array} & \text{Member} \\
[B_0] = & \begin{bmatrix}
0 & 0 \\
-0.667 & +0.500 \\
-0.500 & -0.375 \\
0 & 0 \\
+0.667 & 0.500 \\
0 & 0 \\
-0.833 & +0.625 \\
+0.833 & +0.625 \\
-0.500 & -0.375 \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{array}{c} AB \\ BC \\ CF \\ FE \\ ED \\ DA \\ DB \\ CE \\ BE \\ AE \\ BF \end{array}
\end{array}
$$

Similarly to determine the $[B_1]$ matrix the redundants X_1, X_2 and X_3 are set unit values one at a time with all other forces including the applied loads held at zero. The internal forces in each case are

$$
\begin{array}{ccc}
& \begin{array}{ccc} X_1 = 1 & X_2 = 1 & X_3 = 1 \end{array} & \text{Member} \\
[B_1] = & \begin{bmatrix}
-0.80 & 0 & 1.0 \\
0 & -0.8 & +0.5 \\
0 & -0.6 & +0.375 \\
0 & -0.8 & 0 \\
-0.8 & 0 & -0.5 \\
-0.6 & 0 & 0 \\
1.0 & 0 & -0.625 \\
0 & 1.0 & -0.625 \\
-0.6 & -0.6 & 0.375 \\
1.0 & 0 & 0 \\
0 & 1.0 & 0
\end{bmatrix} & \begin{array}{c} AB \\ BC \\ CF \\ FE \\ ED \\ DA \\ DB \\ CE \\ BE \\ AE \\ BF \end{array}
\end{array}
$$

The flexibility matrix for the members is

$$
[F] = \frac{1}{EA} \begin{bmatrix}
2 &&&&&&&&&& \\
& 2 &&&&&&&&& \\
&& 3 &&&&&&&& \\
&&& 4 &&&&&&& \\
&&&& 4 &&&&&& \\
&&&&& 3 &&&&& \\
&&&&&& 5 &&&& \\
&&&&&&& 5 &&& \\
&&&&&&&& 3 && \\
&&&&&&&&& 5 & \\
&&&&&&&&&& 5
\end{bmatrix}
$$

From [7.36]

$$[H] = [B_0 \mid B_1]^{\mathrm{T}} [F] [B_0 \mid B_1]$$

substituting and carrying out the matrix multiplications gives

$$[H] = \frac{1}{EA} \left[\begin{array}{cc|ccc} 11.111 & 1.792 & -5.400 & 7.031 & -3.125 \\ 1.792 & 6.25 & 2.200 & 3.675 & -5.250 \\ \hline -5.400 & 2.200 & 16.000 & 1.080 & -3.800 \\ 7.031 & 3.675 & 1.080 & 16.000 & -5.275 \\ -3.125 & -5.250 & -3.800 & -5.275 & 8.250 \end{array}\right]$$

$$= \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right]$$

The redundants are determined using [7.40]

$$[X] = \begin{bmatrix} 16.000 & 1.080 & -3.800 \\ 1.080 & 16.000 & -5.275 \\ -3.800 & -5.275 & 8.250 \end{bmatrix}^{-1} \begin{bmatrix} 5.400 & 2.200 \\ 7.031 & 3.675 \\ -3.125 & -5.250 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Solving for the redundants,

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4.617 \\ -3.743 \\ 9.883 \end{bmatrix}$$

The bar forces are determined using [7.31]

$$\begin{bmatrix} N_{\mathrm{AB}} \\ N_{\mathrm{BC}} \\ N_{\mathrm{CF}} \\ N_{\mathrm{FE}} \\ N_{\mathrm{ED}} \\ N_{\mathrm{DA}} \\ N_{\mathrm{DB}} \\ N_{\mathrm{CE}} \\ N_{\mathrm{BE}} \\ N_{\mathrm{AE}} \\ N_{\mathrm{BF}} \end{bmatrix} = \begin{bmatrix} 6.197 \\ 6.265 \\ -2.799 \\ 2.997 \\ 3.054 \\ -2.759 \\ -3.654 \\ 4.661 \\ -5.558 \\ 4.598 \\ 3.747 \end{bmatrix}$$

EXAMPLE 7.2 *Determine the shear force and bending moment values in the continuous beam of Fig. 7.8 using the flexibility matrix method.*

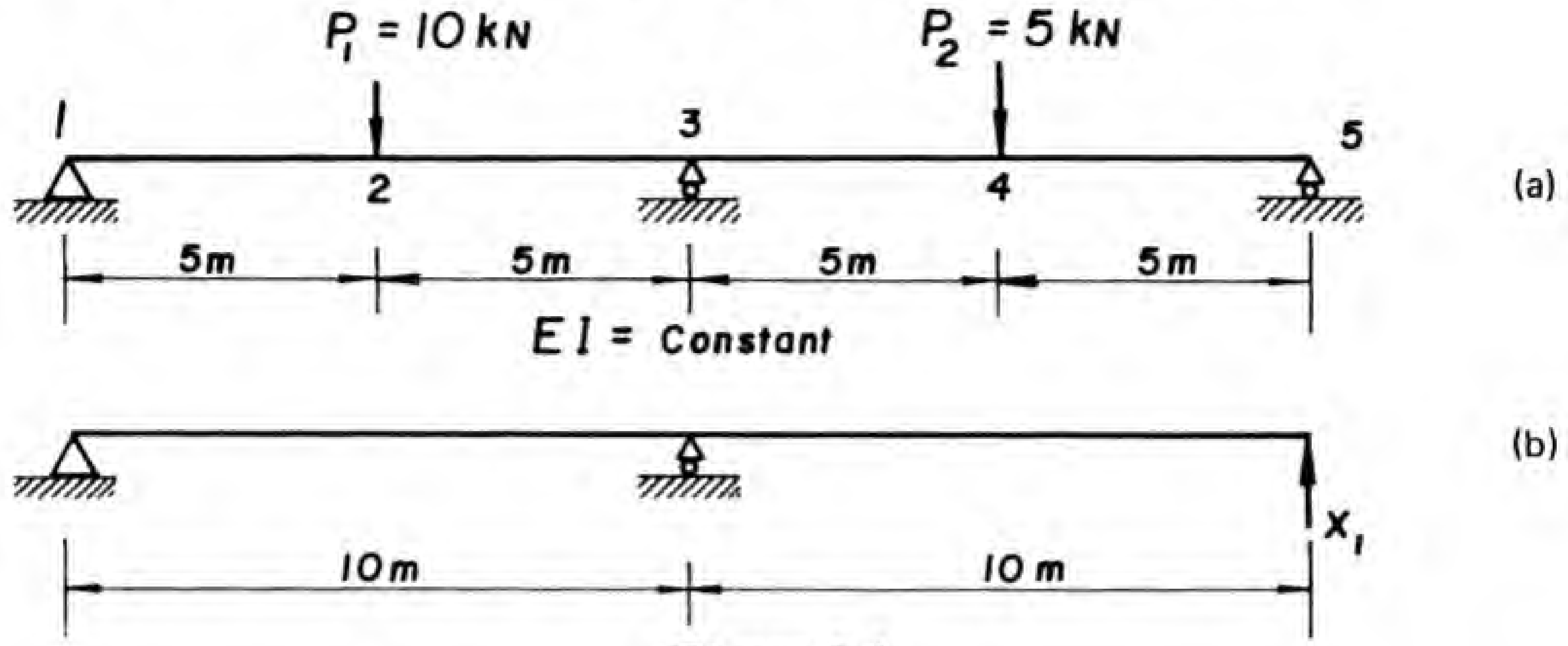


Figure 7.8

The beam is indeterminate to the first degree and the reaction at mode 5 is chosen as the redundant as shown in Fig. 7.8(b).

To determine the $[B_0]$ matrix, P_1 and P_2 are set unit values one at a time, thus

$$[B_0] = \begin{array}{c} \begin{array}{cc} P_1 = 1 & P_2 = 1 \end{array} \\ \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0 \\ -0.5 & -0.5 \\ 2.5 & -2.5 \\ 0 & 1 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

Similarly the $[B_1]$ matrix is determined setting $X_1 = 1$, thus

$$[B_1] = \begin{array}{c} X_1 = 1 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \\ -1 \\ 10 \\ -1 \\ 5 \end{bmatrix} \end{array} \begin{array}{c} \\ V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{array}$$

The member flexibility matrix from [7.20] neglecting the axial deformation is

$$[F] = \frac{0.833}{EI}\begin{bmatrix} \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & & \\ & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & & \\ & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} & \\ & & & \begin{bmatrix} 50 & 15 \\ 15 & 6 \end{bmatrix} \end{bmatrix}$$

Using [7.36] to solve for the $[H]$ matrix

$$[H] = [B_0 \mid B_1]^{\mathrm{T}}[F][B_0 \mid B_1]$$

$$[H] = \frac{0.833}{EI}\left[\begin{array}{cc|c} 25.0 & -37.5 & 75.0 \\ -37.5 & 150.0 & -325.0 \\ \hline 75.0 & -325.0 & 800.0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} H_{pp} & H_{px} \\ \hline H_{xp} & H_{xx} \end{array}\right]$$

The redundant is determined using [7.40]. Hence,

$$X_1 = -\frac{1}{800}\,[75 \quad -325]\begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$= 1.09375 \text{ kN}$$

The shear force and bending moment values at the indicated nodes are obtained using [7.31].

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \\ V_4 \\ M_4 \end{bmatrix} = \left[\begin{array}{cc|c} 0.5 & -0.5 & 1 \\ 0 & 0 & 0 \\ -0.5 & -0.5 & 1 \\ 2.5 & -2.5 & 5 \\ 0 & 1 & -1 \\ 0 & -5 & 10 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{array}\right]\left[\begin{array}{c} 10 \\ 5 \\ \hline 1.09375 \end{array}\right] = \begin{bmatrix} 3.594 \\ 0 \\ -6.406 \\ 17.969 \\ 3.906 \\ -14.063 \\ -1.094 \\ 5.469 \end{bmatrix}$$

7.4 THE STIFFNESS METHOD

As in the flexibility method, the stiffness method considers a structure as an assemblage of individual members. The connecting members are called *node points*. The fundamental difference in using the stiffness method is that the individual displacements of the nodes are taken as the unknowns. In the stiffness method the number of unknowns to be determined is the same as the degree of kinematic indeterminateness.

In this method, the first step is to derive the stiffness matrix for a component member by relating the member forces to the member deformations. In a similar manner the nodal forces must be related to the nodal displacements by the total stiffness matrix obtained from an assemblage of the stiffness matrices of the individual members. Finally, from equilibrium conditions the nodal forces obtained from the unknown nodal displacements must balance the externally applied nodal forces to find the total solution; that is, determining all unknown displacements, reactions and member forces. In developing the stiffness method, the same coordinate systems are used that were employed in the flexibility method.

Member Stiffnesses

The relationship between the forces acting at the nodes P_i and their corresponding nodal displacements forms the stiffness matrix approach. This relationship is given in matrix notation by [7.14] and in its generalised form by [7.15].

Consider a prismatic axial rod element m the ends of which are denoted as i and j as shown in Fig. 7.9.

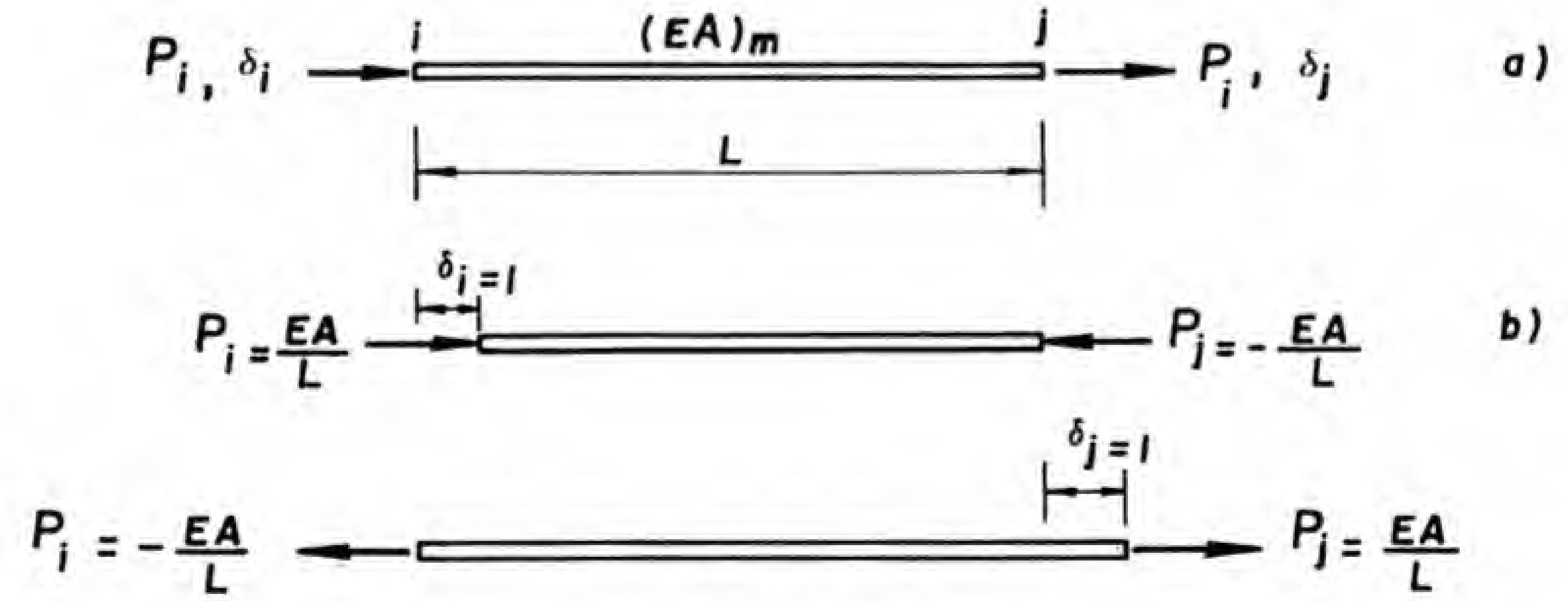


Figure 7.9

The relationship between the axial forces and the corresponding displacements of the rod is

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \qquad [7.41]$$

The coefficients of the stiffness matrix are found by considering two distinct displacement states. The first state is to let the nodal coordinate displacement $\delta_i = 1$ as shown in Fig. 7.9(b) while holding the others at zero. Imposing equilibrium on the forces gives

$$P_i = -P_j = \left(\frac{EA}{L}\right)_m \qquad [7.42]$$

The second state is similar, but distinct from the first. Following the same procedure as in the first state provides

$$P_j = -P_i = \left(\frac{EA}{L}\right)_m \qquad [7.43]$$

Combining the results given by [7.42] and [7.43] into a single matrix equation yields the force–displacement relationship of an axial rod element

$$\begin{bmatrix} P_i \\ P_j \end{bmatrix} = \left(\frac{EA}{L}\right)_m \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} \qquad [7.44]$$

Consider a prismatic beam element shown in Fig. 7.10. Using the same procedure used in obtaining [7.23] to [7.25] the force–displacement relationship for the given nodal coordinate system may be determined by assigning unit values to the displacements as shown in Fig. 7.10(b) and (c). The coefficients are shown for unit displacements at the end i; similar

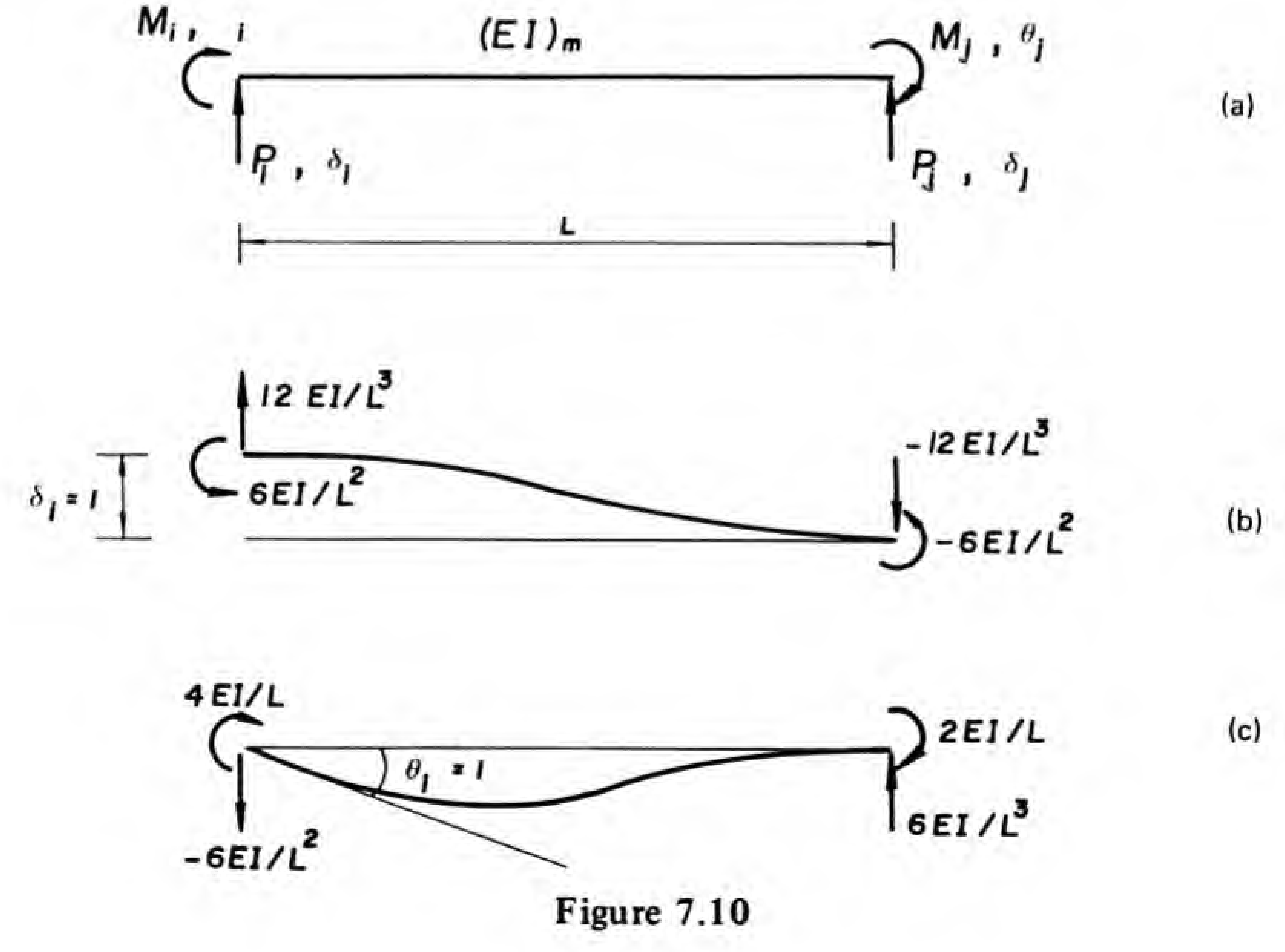


Figure 7.10

coefficients are obtained for unit displacements at end j. Thus

$$\begin{bmatrix} P_i \\ M_i \\ P_j \\ M_j \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_i \\ \theta_i \\ \delta_j \\ \theta_j \end{bmatrix} \qquad [7.45]$$

Transformation Matrices

If the properties of an element is known in terms of *local* axes, the transformations of these forces and displacements to the *global* coordinates is a necessary step in stiffness matrix formulation. Figure 7.11 shows member ij described by the two coordinate systems. The local coordinates are shown as x' and y' and global coordinates as x and y. In this text, the forces displacements and stiffness matrices with respect to local axes are identified by primes. The prime is omitted when written with respect to global axes.

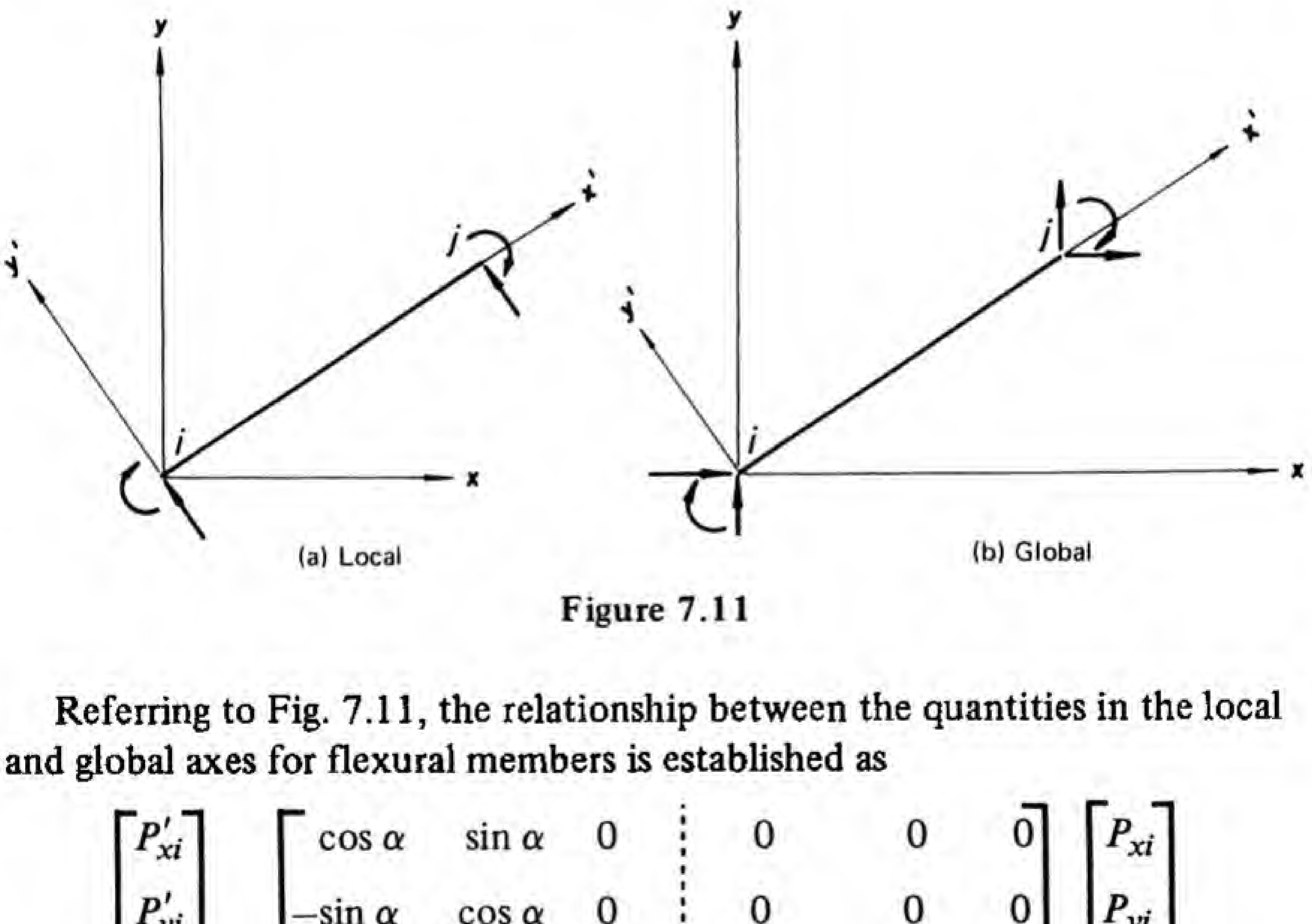


Figure 7.11

Referring to Fig. 7.11, the relationship between the quantities in the local and global axes for flexural members is established as

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ M'_i \\ \hline P'_{xj} \\ P'_{yj} \\ M'_j \end{bmatrix} = \left[\begin{array}{ccc|ccc} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} P_{xi} \\ P_{yi} \\ M_i \\ \hline P_{xj} \\ P_{yj} \\ M_j \end{bmatrix} \qquad [7.46]$$

and for axial members, after omitting M and θ, the relationship is

$$\begin{bmatrix} P'_{xi} \\ P'_{yi} \\ P'_{xj} \\ P'_{yi} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} P_{xi} \\ P_{yi} \\ P_{xj} \\ P_{yj} \end{bmatrix} \qquad [7.47]$$

Equations [7.46] and [7.47] may be written in compact matrix form

$$[P'] = [T]\,[P] \qquad [7.48]$$

where $[T]$ is the *rotational transformation matrix*, which is a function of the direction cosines between the two sets of axes, for the particular system shown. Solving [7.48] for $[P]$

$$\begin{aligned} [P] &= [T^{-1}]\,[P'] \\ &= [T]^{\mathrm{T}}\,[P'] \end{aligned} \qquad [7.49]$$

Such a matrix is called an *orthogonal matrix*, which may be defined as a square matrix having an inverse equal to its transpose.

If the displacements are denoted by $[\delta]$ then it follows that

$$[\delta'] = [T]\,[\delta] \qquad [7.50]$$

The transformation matrix $[T]$ may be applied to obtain the stiffness matrix in global coordinates. From the definition of stiffness, that is $[P] = [k]\,[\delta]$, it follows that

$$[P'] = [k']\,[\delta'] \qquad [7.51]$$

Substituting [7.48] and [7.50] into [7.51] and noting that $T^T = T^{-1}$ for orthogonal matrices

$$[T]\,[P] = [k']\,[T]\,[\delta]$$

or

$$\begin{aligned} [P] &= [T^{-1}]\,[k']\,[T]\,[\delta] \\ &= [T]^T [k']\,[T]\,[\delta] \\ &= [k]\,[\delta] \end{aligned} \qquad [7.52]$$

Hence, the transformed stiffness matrix is given by

$$[k] = [T]^T [k']\,[T] \qquad [7.53]$$

Using the relationship derived above, the stiffness matrix for axial members

(Fig. 7.11) in global coordinates will be

$$[k] = \left(\frac{EA}{L}\right)_m \left[\begin{array}{cc:cc} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ \hdashline -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{array}\right] \qquad [7.54]$$

where $\lambda = \cos\alpha$ and $\mu = \sin\alpha$. Similarly, for flexural members

$$[k] = \left(\frac{EI}{L^3}\right)_m \left[\begin{array}{ccc:ccc} 12\mu^2 & & & & \text{Symmetric} & \\ -12\lambda\mu & 12\lambda^2 & & & & \\ 6L\mu & -6L\lambda & 4L^2 & & & \\ \hdashline -12\mu^2 & 12\lambda\mu & -6L\mu & 12\mu^2 & & \\ 12\lambda\mu & -12\lambda^2 & 6L\lambda & -12\lambda\mu & 12\lambda^2 & \\ 6L\mu & -6L\lambda & 2L^2 & -6L\mu & 6L\lambda & 4L^2 \end{array}\right] \qquad [7.55]$$

Assembly of Element Matrices

It is important to form the total assemblage nodal stiffness matrix of a structure from the stiffness matrices of the separate structural elements. This involves only simple additions when all element stiffness matrices have been expressed in the same global coordinate system.

Consider the axial member system shown in Fig. 7.12 with a total of three possible joint displacements one for each node. The members have individual stiffness constants $(EA/L)_1$ and $(EA/L)_2$ as shown in the figure.

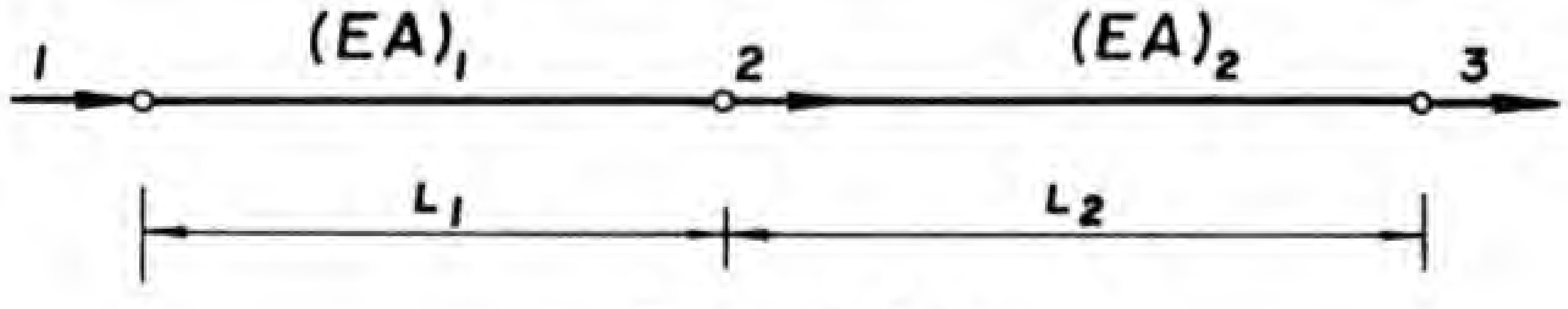


Figure 7.12

The order of the stiffness matrix for the assemblage will be 3 x 3. The individual member stiffness matrices are:

$$[k_1] = \left(\frac{EA}{L}\right)_1 \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{l} 1 \\ 2 \end{array}$$

$$[k_2] = \left(\frac{EA}{L}\right)_2 \begin{array}{c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{l} 3 \\ 4 \end{array} \qquad [7.56]$$

The assembled stiffness matrix for the complete system can be formed by superposition of the individual element stiffnesses contributing to each nodal point. Thus, the assembled stiffness matrix for the system shown in Fig. 7.12 becomes

$$[K] = \begin{array}{c} \left[\begin{array}{c:c:c} \left(\frac{EA}{L}\right)_1 & -\left(\frac{EA}{L}\right)_1 & 0 \\ \hdashline -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_1 + \left(\frac{EA}{L}\right)_2 & -\left(\frac{EA}{L}\right)_2 \\ \hdashline 0 & -\left(\frac{EA}{L}\right)_1 & \left(\frac{EA}{L}\right)_2 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad [7.57]$$

In a similar manner, it may be concluded that the order of the global stiffness matrix of a system is equal to the total number of degrees of freedom of the system. The order of the matrix may be expressed as the sum of the unknown displacements f and the prescribed (support) displacements s. After reordering the rows and columns to separate the elements corresponding to the supports from the remainder, the rearranged stiffness matrix may be written as

$$\begin{bmatrix} P_f \\ \hdashline P_s \end{bmatrix} = \left[\begin{array}{c:c} K_{ff} & K_{fs} \\ \hdashline K_{sf} & K_{ss} \end{array}\right] \begin{bmatrix} \Delta_f \\ \hdashline \Delta_s \end{bmatrix} \qquad [7.58]$$

Method of Solution

Expanding [7.58] and noting that the support displacements, $\{\Delta_s\} = 0$

$$[P_f] = [K_{ff}]\,[\Delta_f] \qquad [7.59(a)]$$

$$[P_s] = [K_{sf}]\,[\Delta_f] \qquad [7.59(b)]$$

The vectors of all unknown nodal displacements (at unsupported nodes) are obtained from [7.59(a)]

$$[\Delta_f] = [K_{ff}]^{-1}\,[P_f] \qquad [7.60]$$

When $[\Delta_f]$ has been found from [7.60], the support reactions by substitution of the results in [7.59(b)] will be

$$[P_s] = [K_{sf}]\,[K_{ff}]^{-1}\,[P_f] \qquad [7.61]$$

The internal force in any element m may be obtained by substituting the calculated degrees of freedom for that element, designated by $[\Delta_m]$, into the element stiffness matrix $[k_m]$. Thus, the joint force component acting on that element becomes

$$[P_m] = [k_m]\,[\Delta_m] \qquad [7.62]$$

For the case of an axial member shown in Fig. 7.11 the member force, denoted by P_m, is found to be

$$P_m = P_{xj} \cos \alpha + P_{yj} \sin \alpha \qquad [7.63]$$

but

$$\begin{aligned} P_{xj} &= \frac{EA}{L} ((\delta_{xj} - \delta_{xi})\cos^2 \alpha + (\delta_{yj} - \delta_{yi})\cos \alpha \sin \alpha) \\ P_{yj} &= \frac{EA}{L} ((\delta_{xj} - \delta_{xi})\cos \alpha \sin \alpha + (\delta_{yj} - \delta_{yi})\sin^2 \alpha) \end{aligned} \qquad [7.64]$$

Rearranging and writing in matrix form, [7.64] may be written as

$$P_{\mathrm{m}} = \left(\frac{EA}{L}\right)_{\mathrm{m}} [\cos \alpha \quad \sin \alpha] \begin{bmatrix} \delta_{xj} - \delta_{xi} \\ \delta_{yj} - \delta_{yi} \end{bmatrix} \qquad [7.65]$$

Similarly, for the case of a beam element [7.45] and the true nodal displacements are used and the internal forces for the beam element taken to be the shear and bending moments are

$$\begin{bmatrix} V_i \\ M_i \end{bmatrix} = \left(\frac{EI}{L^3}\right)_m \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \end{bmatrix} \begin{bmatrix} \delta_{yi} \\ \theta_i \\ \delta_{yi} \\ \theta_j \end{bmatrix} \qquad [7.66]$$

When these are external loads acting between the joints of a beam element the concept of equivalent loads must be adopted. The member action is then computed by adding the effects of the member end deformation to the fixed-end actions produced by the loads. In a similar manner, the support reactions are computed by adding the fixed-end effects of the loads. Thus [7.14] may be written as

$$[P] = [K]\,[\Delta] - [P^F] \qquad [7.67]$$

where $[P^F]$ is the load vector of the fixed-end actions.

EXAMPLE 7.3 *Determine the bar forces, using the stiffness matrix method, of the truss shown in Fig. 7.13. EA = constant.*

Member data for the truss

Member	Member ends		Member properties		*Direction cosines*	
	i	j	A	L	*cos* α	*sin* α
1	A	B	A	L	0	1
2	A	C	A	$1.155L$	–0.5	–0.866
3	A	D	A	$1.4146L$	0.707	–0.707

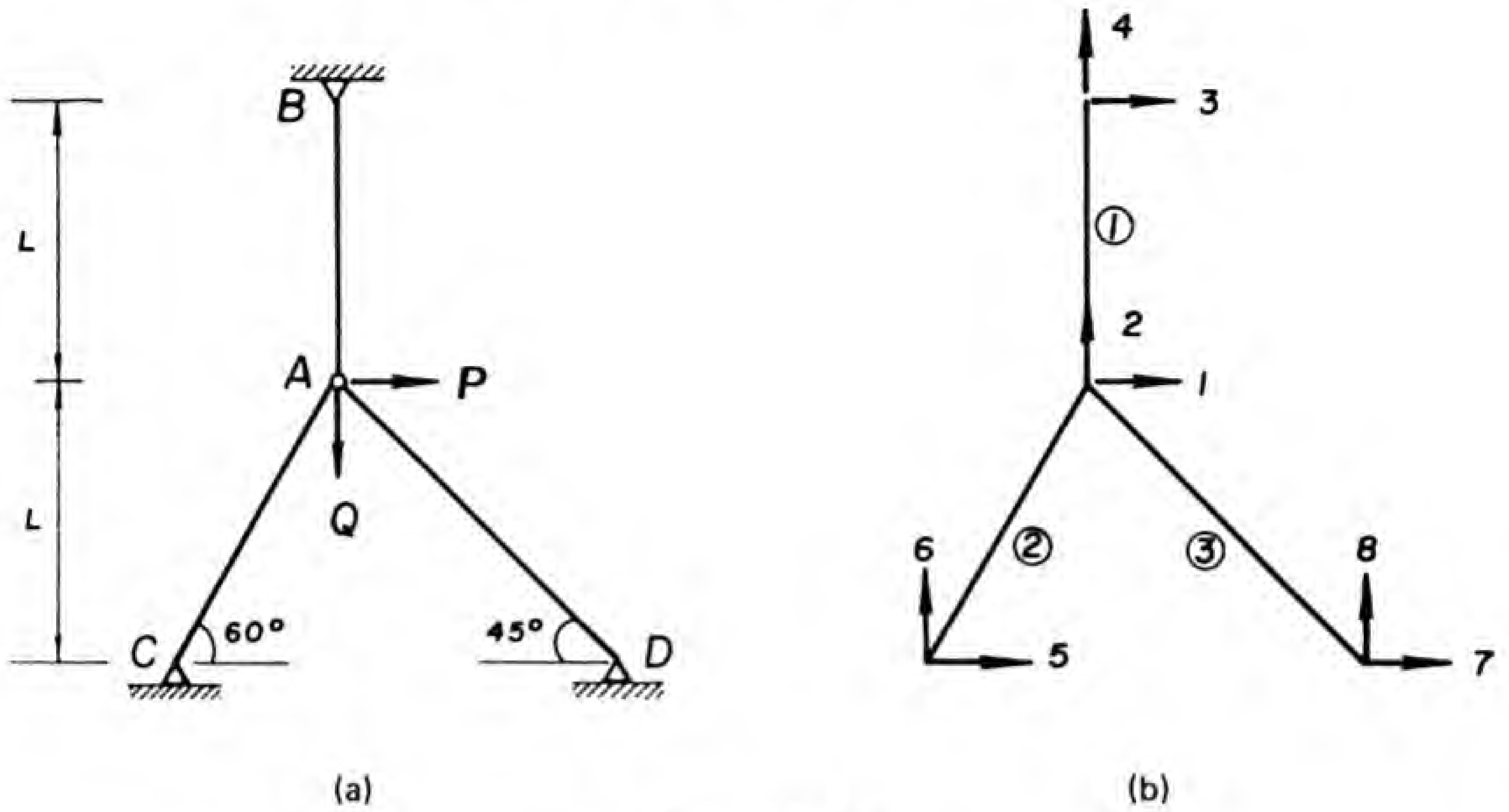


Figure 7.13

The member stiffnesses oriented in global coordinate system are obtained from [7.54].

Member AB

$$[k_1] = \frac{EA}{L}\begin{array}{c}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Member AC

$$[k_2] = \frac{EA}{1.155L}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$k_3 = \frac{EA}{1.414L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

After assemblage of the element stiffness matrices, the global stiffness

equation becomes

$$\begin{bmatrix} P \\ -Q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{EA}{L} \left[\begin{array}{cc:cccccc} 0.570 & 0.021 & 0 & 0 & -0.217 & -0.375 & -0.354 & 0.354 \\ 0.021 & 2.00 & 0 & -1.0 & -0.375 & -0.650 & 0.354 & -0.354 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 1.0 & 0 & 0 & 0 & 0 \\ -0.217 & -0.375 & 0 & 0 & 0.217 & 0.375 & 0 & 0 \\ -0.375 & -0.650 & 0 & 0 & 0.375 & 0.650 & 0 & 0 \\ -0.354 & 0.354 & 0 & 0 & 0 & 0 & 0.354 & -0.354 \\ 0.354 & -0.354 & 0 & 0 & 0 & 0 & -0.354 & 0.354 \end{array}\right] \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \end{bmatrix}$$

Note that the displacements Δ_3 to Δ_8 are restrained and the elements corresponding to Δ_1 and Δ_2 are placed at the top left of the global stiffness matrix. Hence the reduced stiffness matrix given by [7.59(a)] is written as

$$\begin{bmatrix} P \\ -Q \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.570 & 0.021 \\ 0.021 & 2.00 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}$$

The unknown displacements are obtained by applying [7.60],

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{L}{EA} \begin{bmatrix} 1.755 & -0.018 \\ -0.018 & 0.500 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

Having found the unknown displacements, the reactions are calculated by applying [7.61]. Thus

$$\begin{bmatrix} P_{Bx} \\ P_{By} \\ P_{Cx} \\ P_{Cy} \\ P_{Dx} \\ P_{Dy} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1.0 \\ -0.217 & -0.375 \\ -0.375 & -0.650 \\ -0.354 & 0.354 \\ 0.354 & -0.354 \end{bmatrix} \begin{bmatrix} 1.755 & -0.018 \\ -0.018 & 0.500 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0.018 & -0.500 \\ -0.373 & -0.184 \\ -0.646 & 0.318 \\ -0.628 & 0.183 \\ 0.628 & -0.183 \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix}$$

The internal member forces are obtained by using Eq. [7.65]. Thus

$$P_m = \frac{EA}{L}[\cos\alpha \quad \sin\alpha]\begin{bmatrix}\Delta_{xj}-\Delta_{xi}\\ \Delta_{yj}-\Delta_{yi}\end{bmatrix}$$

Since $\Delta_{xj} = \Delta_{yj} = 0$ because of support conditions, the member forces are:

Member AB

$$P_{AB} = [0 \quad 1]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018P + 0.50Q\end{bmatrix}$$

$$= 0.018P + 0.50Q$$

Member AC

$$P_{AC} = \frac{1}{1.155}[-0.5 \quad -0.866]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018P + 0.50Q\end{bmatrix}$$

$$= 0.746P + 0.367Q$$

Member AD

$$P_{AD} = \frac{1}{1.414}[0.707 \quad -0.707]\begin{bmatrix}-1.755P - 0.018Q\\ 0.018Q + 0.50Q\end{bmatrix}$$

$$= -0.887P - 0.259Q$$

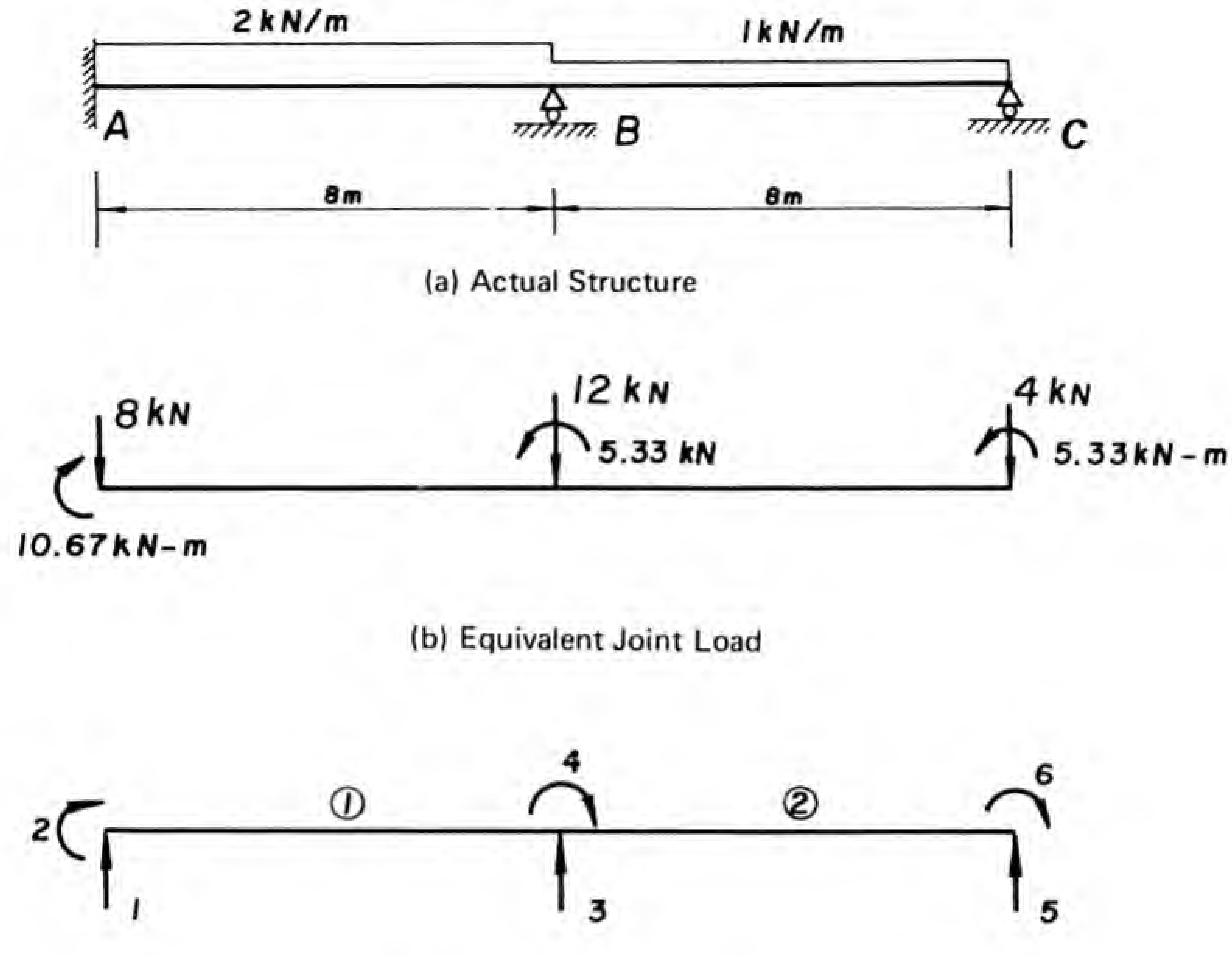
EXAMPLE 7.4 *Find the support rotations and the support reactions of the continuous beam shown in Fig. 7.14.*

The equivalent fixed-end actions are shown in Fig. 7.14(b).

The member stiffnesses are (see [7.45]):

Member AB

$$[k_1] = \frac{EI}{512}\begin{array}{c}\begin{bmatrix}12 & -48 & -12 & -48\\ -48 & 256 & 48 & 128\\ -12 & 48 & 12 & 48\\ -48 & 128 & 48 & 256\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}$$



(a) Actual Structure

(b) Equivalent Joint Load

(c) Coordinate System

Figure 7.14

Member BC

$$[k_2] = \frac{EI}{512} \begin{array}{c} \begin{bmatrix} 12 & -48 & -12 & -48 \\ -48 & 256 & 48 & 128 \\ -12 & 48 & 12 & 48 \\ -48 & 128 & 48 & 256 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

The assembled stiffness matrix is

$$[K] = \frac{EI}{512} \begin{bmatrix} 12 & -48 & -12 & -48 & 0 & 0 \\ -48 & 256 & 48 & 128 & 0 & 0 \\ -12 & 48 & 24 & 0 & -12 & -48 \\ -48 & 128 & 0 & 512 & 48 & 128 \\ 0 & 0 & -12 & 48 & 12 & 48 \\ 0 & 0 & -48 & 128 & 48 & 256 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Note that the prescribed displacements are $\delta_1 = \delta_2 = \delta_5 = 0$ and δ_4 and δ_6 are the unknown displacements. Thus the assembled stiffness matrix must be

rearranged by moving the rows and columns associated to δ_4 and δ_6 to the top left of the stiffness matrix. Thus

$$[K] = \frac{EI}{512}\left[\begin{array}{cccccc} 512 & 128 & -48 & 128 & 0 & 48 \\ 128 & 256 & 0 & 0 & -48 & 48 \\ -48 & 0 & 12 & -48 & -12 & 0 \\ 128 & 0 & -48 & 256 & 48 & 0 \\ 0 & -48 & -12 & 48 & 24 & 12 \\ 48 & 48 & 0 & 0 & -12 & 12 \end{array}\right]\begin{array}{c} 4 \\ 6 \\ 1 \\ 2 \\ 3 \\ 5 \end{array}$$

$$= \left[\begin{array}{c:c} K_{ff} & K_{fs} \\ \hdashline K_{sf} & K_{ss} \end{array}\right]$$

The submatrix $[K_{ff}]$ is

$$[K_{ff}] = \frac{EI}{512}\begin{bmatrix} 512 & 128 \\ 128 & 256 \end{bmatrix}$$

and its inverse is

$$[K_{ff}]^{-1} = \frac{1}{224EI}\begin{bmatrix} 256 & -128 \\ -128 & 512 \end{bmatrix}$$

The load vector for the beam shown in Fig. 7.14(b) is

$$[P] = \begin{bmatrix} -8 \\ 10.67 \\ -12 \\ -5.33 \\ -4 \\ -5.33 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Rearranging the vector in conformity with the stiffness matrix is

$$[P] = \left[\begin{array}{c} -5.33 \\ -5.33 \\ \hdashline -8 \\ 10.67 \\ -12 \\ -4 \end{array}\right]\begin{array}{c} 4 \\ 6 \\ 1 \\ 2 \\ 3 \\ 5 \end{array}$$

The unknown rotations at support B and C are computed using [7.60]

$$[\Delta_f] = \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = [K_{ff}]\,[P_f]$$

$$= \frac{1}{224EI}\begin{bmatrix} 256 & -128 \\ -128 & 512 \end{bmatrix}\begin{bmatrix} -5.33 \\ -5.33 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -3.05 \\ -9.14 \end{bmatrix}$$

The support reactions due to the displacement contributions are determined by using [7.59(b)]. After adding the fixed-end effects of the loads, the support reactions will be

$$[P_s] = K_{sf}[\Delta_f] - [P^F]$$

$$\begin{bmatrix} R_A \\ M_A \\ R_B \\ R_C \end{bmatrix} = \frac{EI}{512}\begin{bmatrix} -48 & 0 \\ 128 & 0 \\ 0 & -48 \\ 48 & 48 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} -3.05 \\ -9.14 \end{bmatrix} - \begin{bmatrix} -8 \\ 10.67 \\ -12 \\ -4 \end{bmatrix} = \begin{bmatrix} 8.29 \\ -11.43 \\ 12.85 \\ 2.86 \end{bmatrix}$$

7.5 PROBLEMS

7.1 Using the flexibility method, find the internal forces on all members for the truss shown in Fig. P7.1. The cross sectional area of the members, in square centimetres are shown in parentheses.

(*Ans:* $F_{AB} = 1.37$ kN $\quad F_{BC} = 5.33$ kN $\quad F_{CE} = 5.94$ kN

$F_{AC} = -1.6$ kN $\quad F_{EF} = -1.98$ kN $\quad F_{FE} = -7.46$ kN

$F_{BE} = -11.47$ kN $\quad F_{BF} = -4.95$ kN $\quad F_{EC} = 0.77$ kN

$F_{CF} = -1.20$ kN)

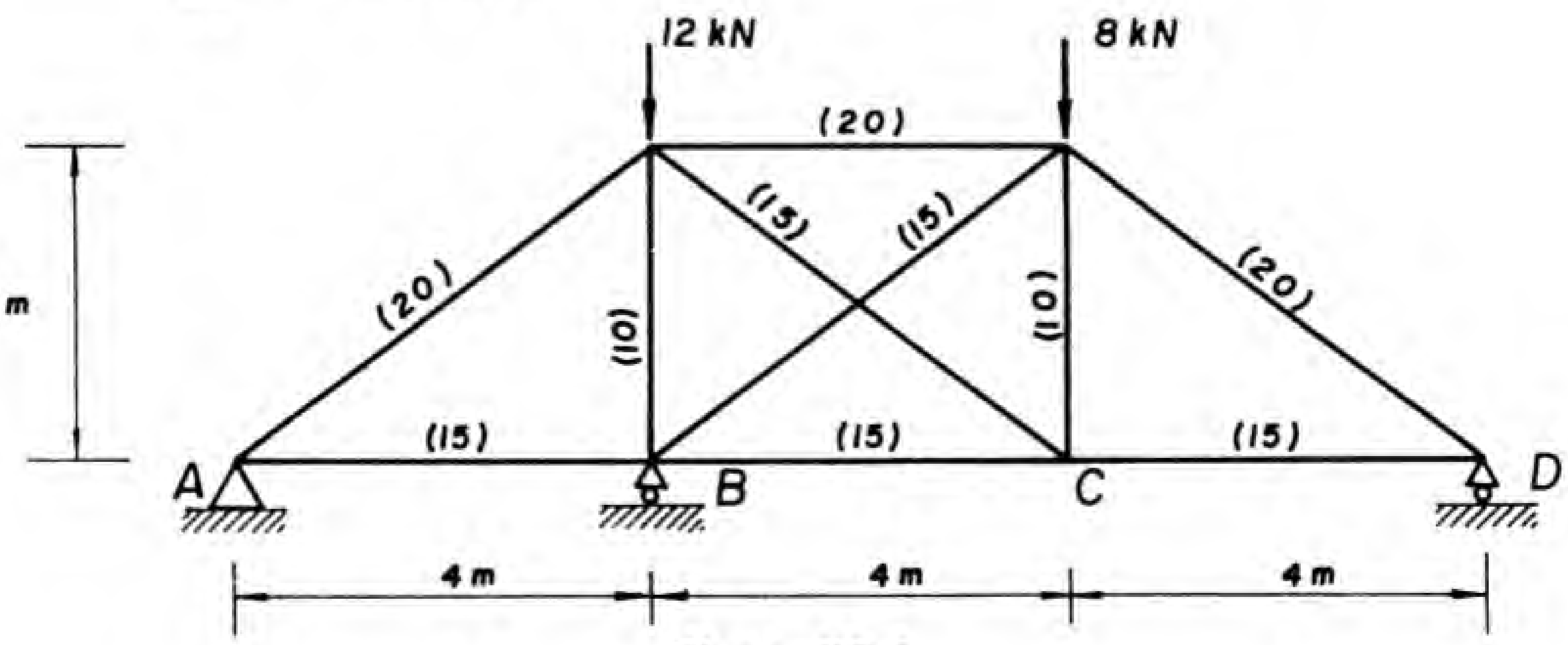


Figure P7.1

7.2 Find the internal forces on all members of the truss shown in Fig. P7.2 using the flexibility method. Assume EA to be constant.

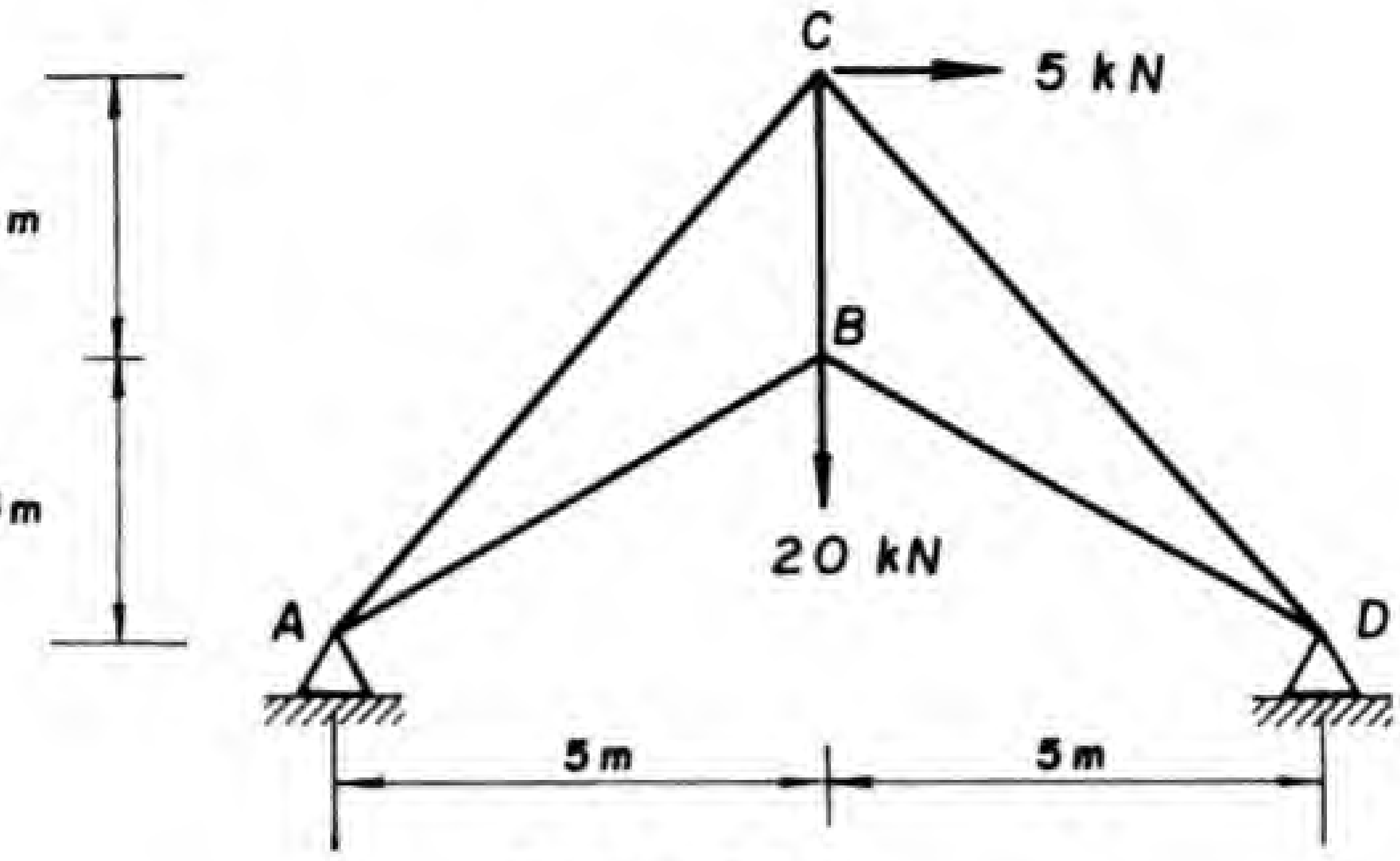


Figure P7.2

7.3 Using the flexibility method, determine the support moments of the continuous beam shown in Fig. P7.3. Assume EI to be constant.

(*Ans*: $M_A = -10.8$ kN m $M_B = -5.4$ kN m $M_C = +2.7$ kN m)

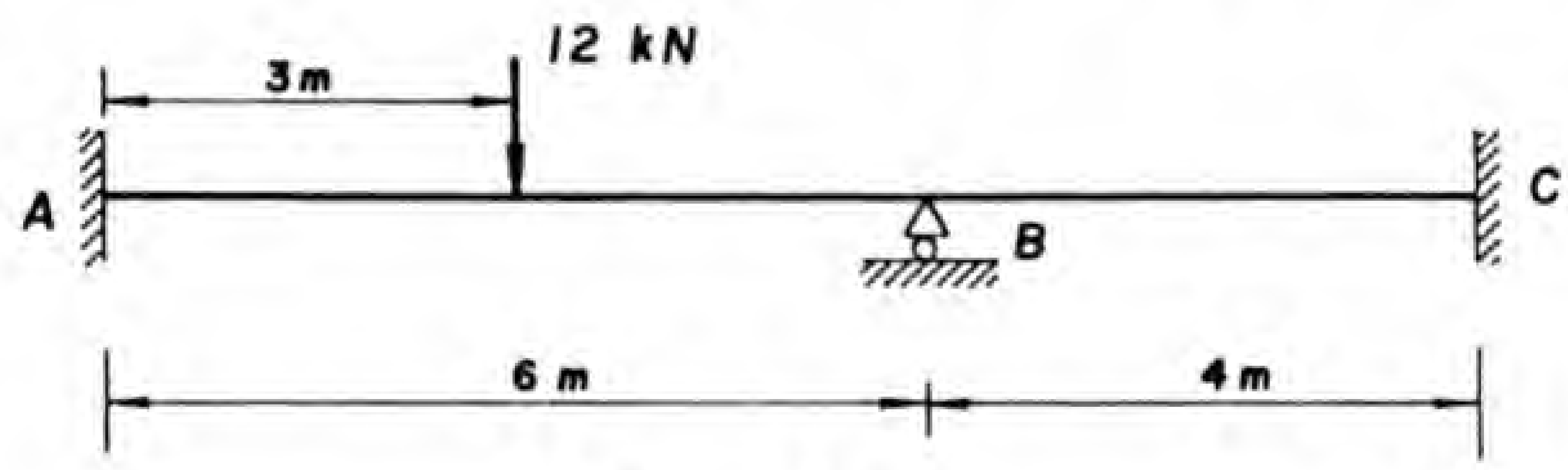


Figure P7.3

7.4 Find the internal forces of the frame shown in Fig. P7.4. Use the flexibility method. Assume EI to be constant.

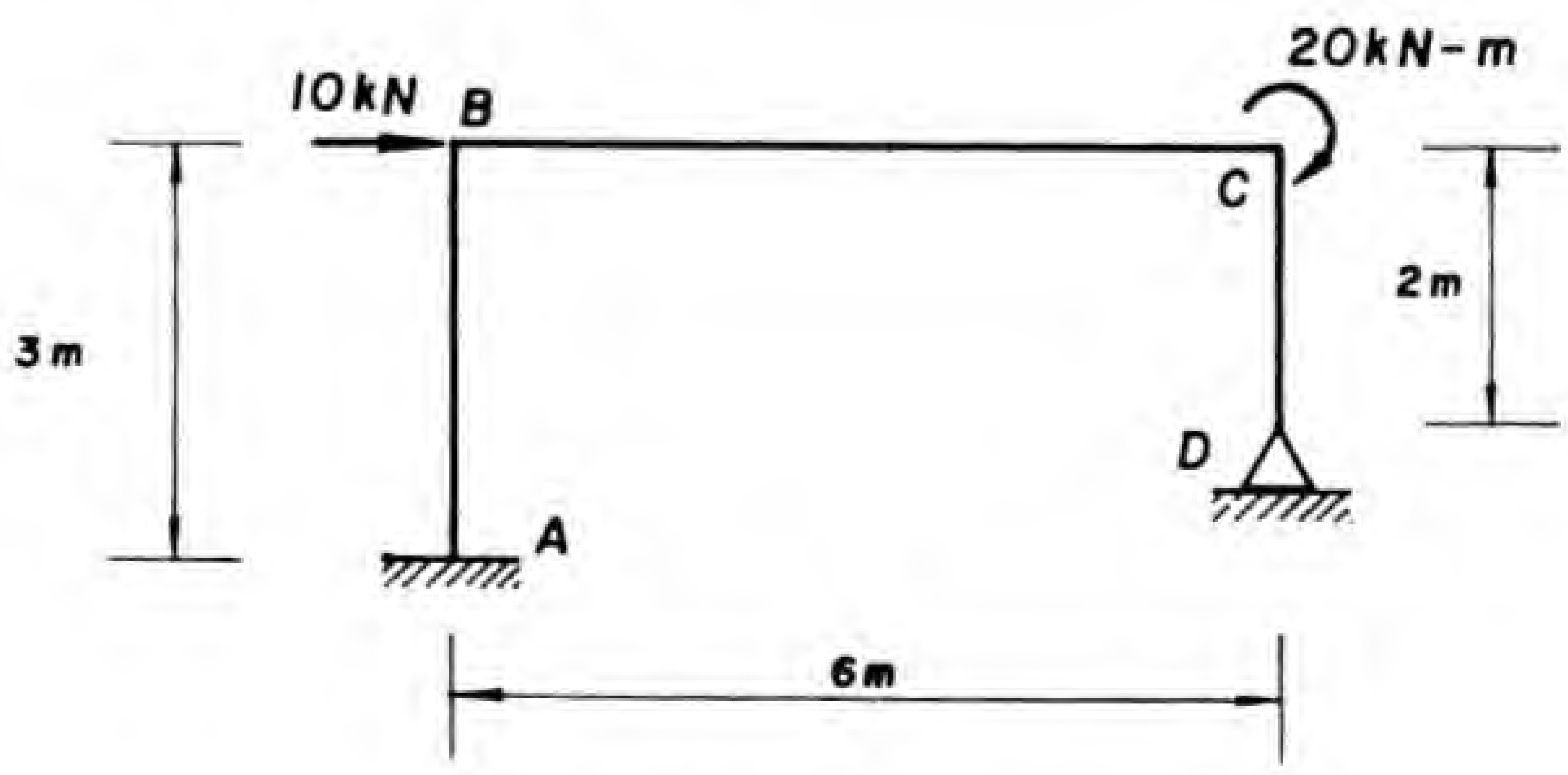


Figure P7.4

7.5 Using the stiffness method, find the internal forces in all members for the truss shown in Fig. P7.5. Assume EA to be constant.

(*Ans*: $F_{AC} = 6.22$ kN $\quad F_{CD} = 4.67$ kN $\quad F_{AD} = 8.90$ kN
$F_{BC} = -7.78$ kN $\quad F_{BD} = -7.10$ kN)

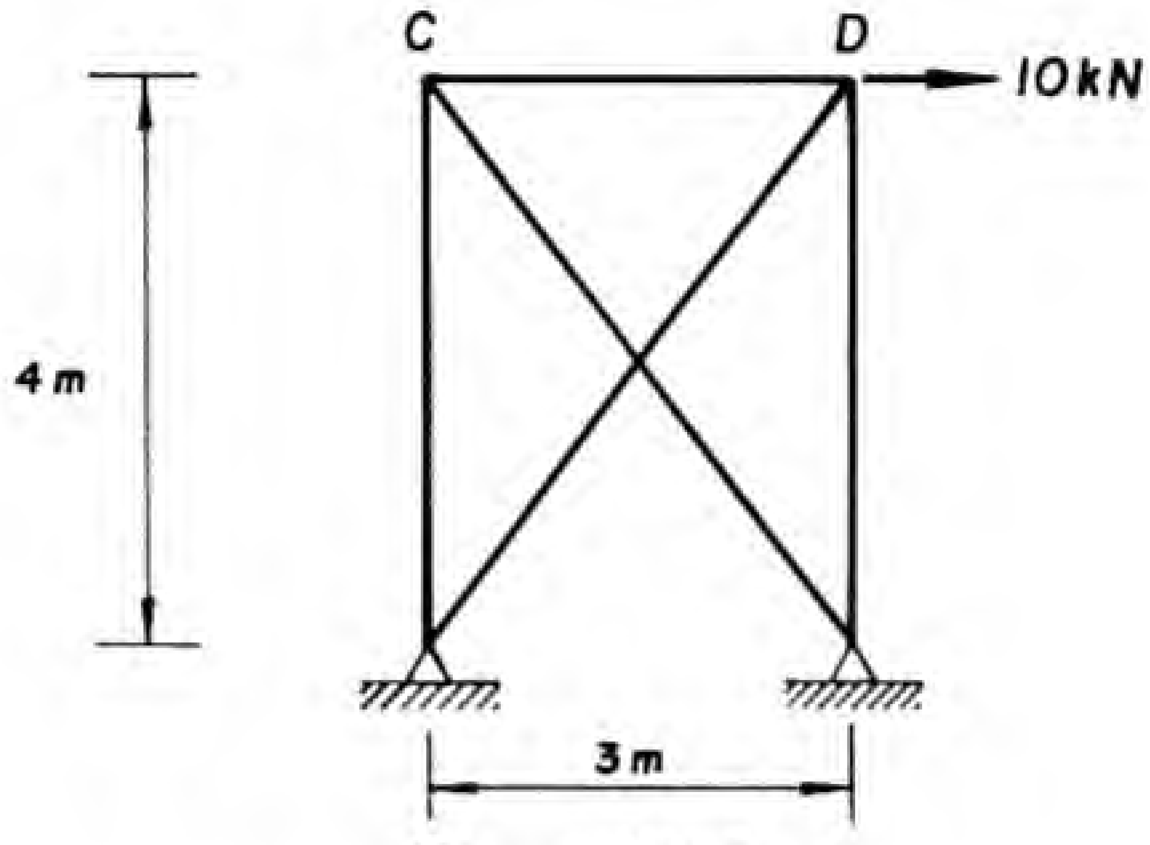


Figure P7.5

7.6 Solve Prob. 7.2 using the stiffness method.

7.7 Determine the support moments for the continuous beam in Fig. P7.7. Assume EI to be constant.

(*Ans*: $M_A = -11.11$ kN m $\quad M_B = -4.77$ kN m)

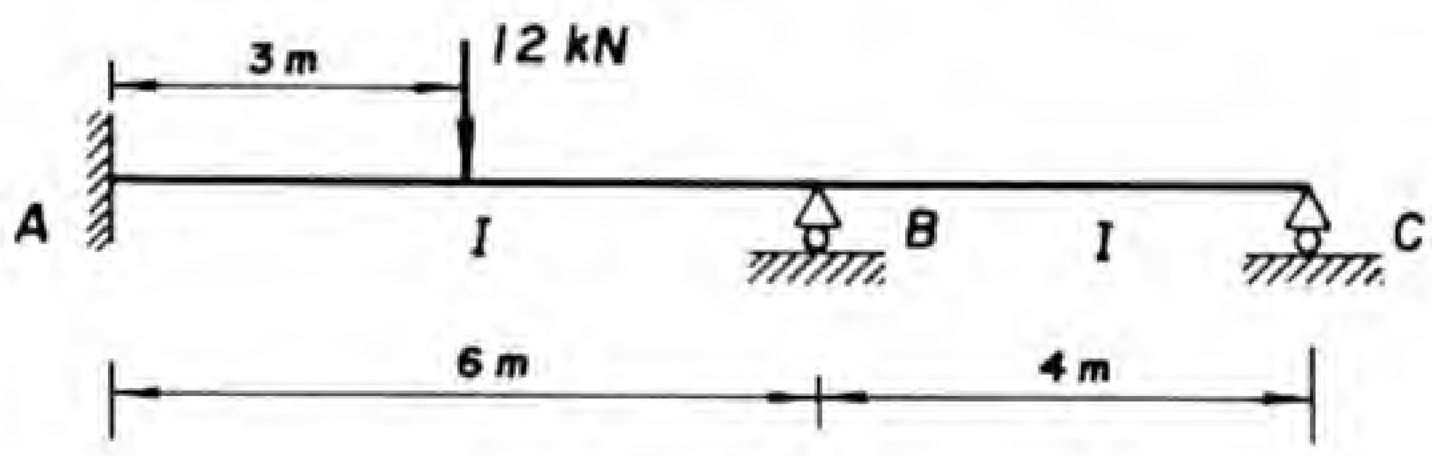


Figure P7.7

7.8 Find the joint moments and shears of the frame shown in Fig. P7.8. Neglect axial deformations.

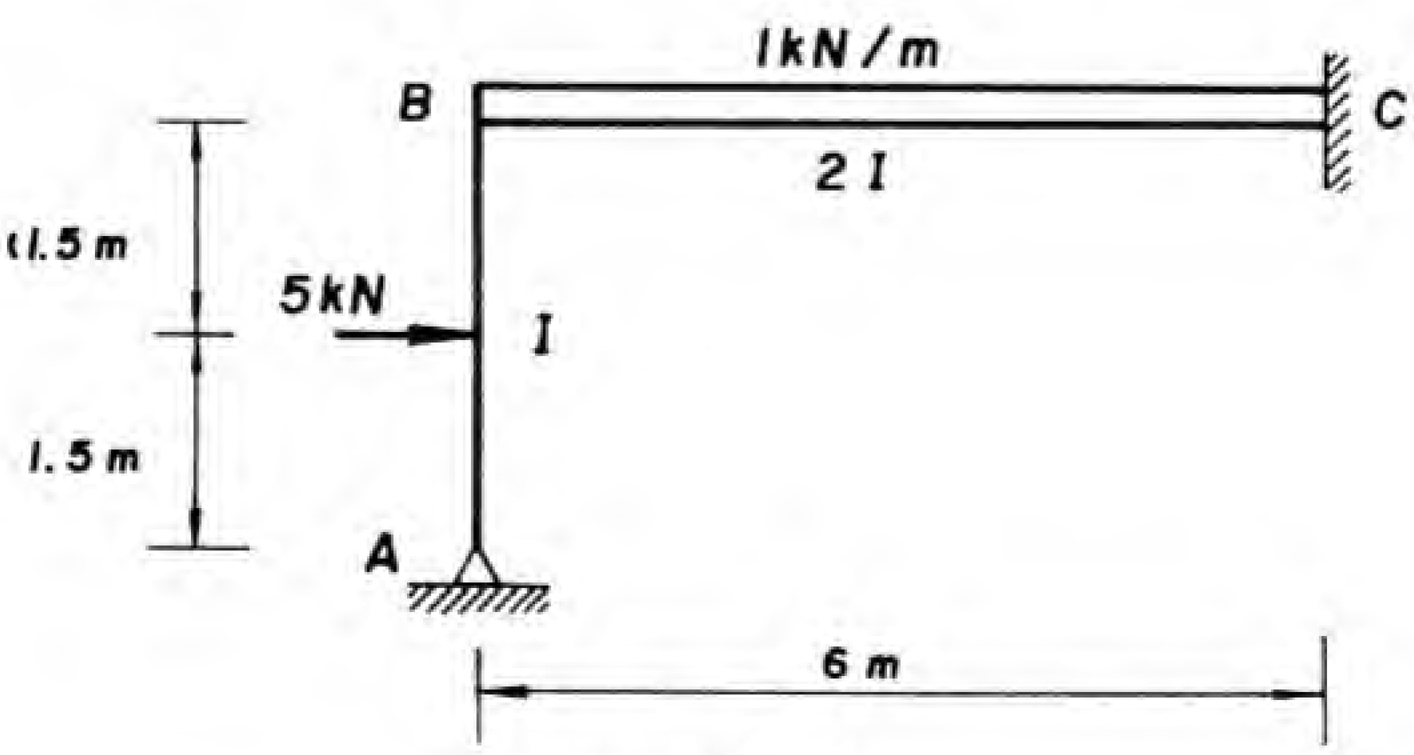


Figure P7.8

7.9 Determine the joint moments of the frame shown in Fig. P7.9. Neglect axial deformations.

(*Ans*: $M_A = 2.59$ kN m $\quad M_B = -5.19$ kN m $\quad M_{CB} = -8.83$ kN m

$M_{CD} = 2.76$ kN m $\quad M_{CE} = -6.07$ kN m $\quad M_{DC} = 1.39$ kN m)

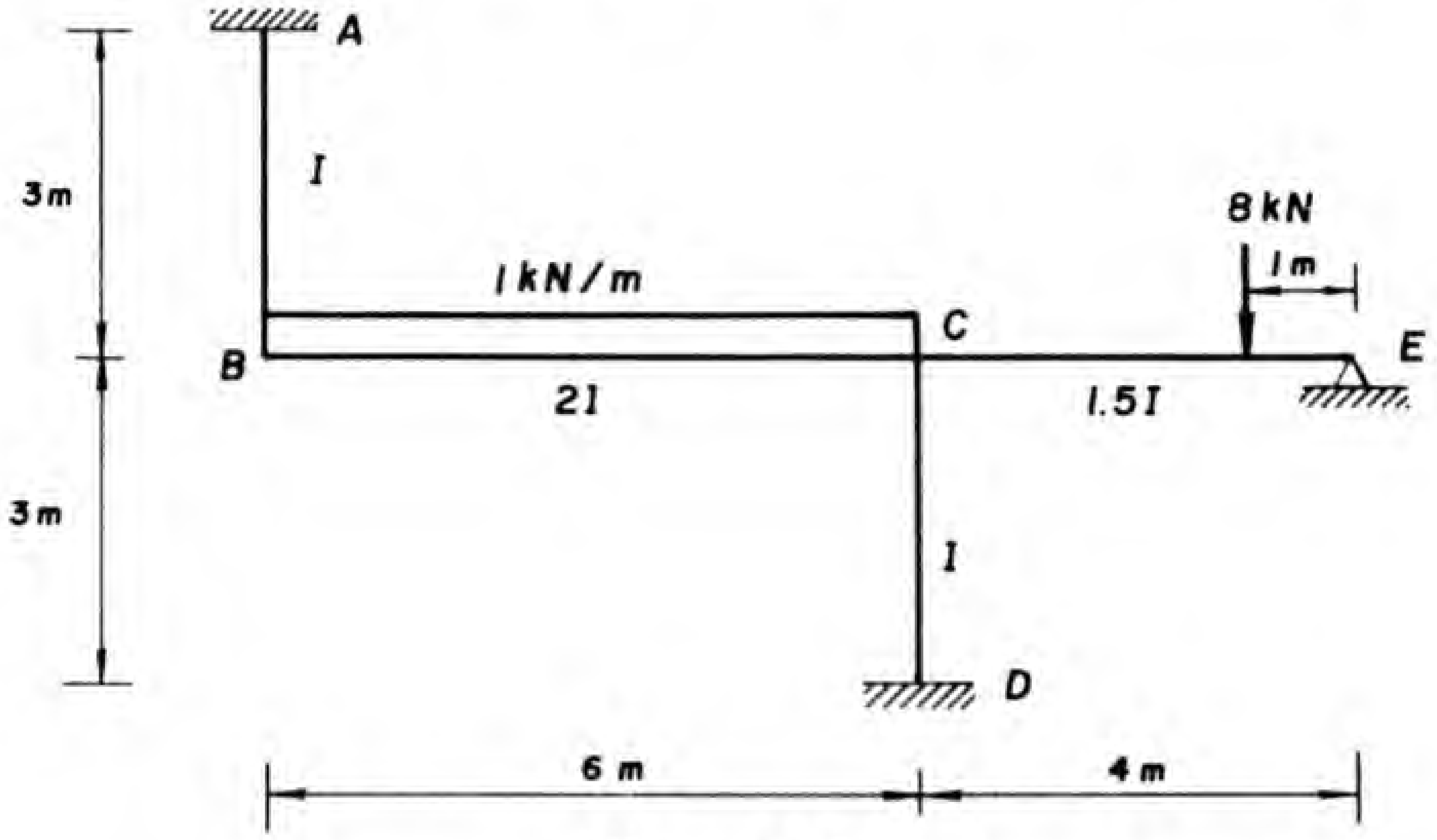


Figure P7.9

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